Higher invariants in homotopy theory

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Talk plan

1. Motivations

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2. Classical Massey products

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3. Interlude: Algebraic operads and Koszul duality

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4. Massey products for algebras over algebraic operads (jt. with J. Moreno-Fernandez [6])

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3. Interlude: Algebraic operads and Koszul duality
4. Massey products for algebras over algebraic operads (jt. with J. Moreno-Fernandez [6])
5. Applications of cotriple products [5]

Motivations and recollections

Motivation: Distinguishing spaces

Question

When are two topological spaces[a] homotopy equivalent?

[a] locally compact, Hausdorff with the homotopy type of a CW-complex

Source: reddit.com

Refresher: Homotopy equivalence of spaces

Definition

Two continuous maps $f, g : X \to Y$ are homotopic if there is a map $H : X \times I \to Y$ such that $H_0 = f$ and $H_1 = g$

Definition

Let X and Y be spaces. They are homotopy equivalent if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity.

Algebras

Definition

A chain complex over a field is a collection of vector spaces $A^i : i \in \mathbb{Z}$ and maps $d^n : A^n \to A^{n+1}$ such that $d^{n+1} \circ d^n = 0$ for all n.

$$\cdots A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} A^{i+2} \xrightarrow{d} A^{i+3} \xrightarrow{d} A^{i+4} \xrightarrow{d} \cdots$$

Definition

A dg-algebra is a chain complex A equipped with a binary associative multiplication $- \cup - : A^p \otimes A^q \to A^{p+q}$ and d is a derivation wrt. $\cup$

$$d(x \cup y) = d(x) \cup y + (-1)^{|x|} x \cup d(y)$$

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Example: if you have a smooth manifold M, the de Rham forms $(\Omega^\bullet(M), \wedge)$

Singular (co)chains

- There is a dg-algebra with $\mathbb{Z}$-coefficients associated to every space[1].

$$C^i(X;\mathbb{Z}) = \text{Hom}\left(\{f : \triangle^n \to X; f \text{ continuous function}\}, \mathbb{Z}\right)$$

Here, $\triangle^n$ is the n-dimensional simplex.

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- The linear dual d_i of the differential d^* is defined on the dual basis takes a function $f : \triangle^n \to X$ to $\sum(-1)^i f_i$ where $f_i : \triangle^{n-1} \to X$ is the i^{th} boundary simplex.

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- $C^*(X;\mathbb{k})$ has multiplication induced by the diagonal map

$$X \to X \times X$$

$$x \mapsto (x, x)$$

[1] of finite type

Cohomology 1

Definition

The cohomology of a chain complex A is the following set

$$H^i(A) = \ker d^i / \operatorname{Im} d^{i-1}$$

- These inherit the addition from A making them groups.
- If A is a dg-algebra, they inherit the multiplication giving them a ring structure.
- If $A = C^*(X : \mathbb{k})$; this is a commutative ring.

Cohomology 2

Theorem (Poincaré circa 1900 and Alexander, Čech, Whitney, Eilenberg circa 1950)

Let X and Y be homotopy equivalent spaces. Then they have the same cohomology rings. Homotopic maps have the same induced map on cohomology.

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The spaces $\mathbb{C}P^\infty \times S^1/\{x_0\} \times S^1$ and $\mathbb{C}P^\infty \times S^3$ have the same cohomology ring but are not homotopy equivalent.

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Motivation: We need stronger invariants

Weak equivalence of algebras

Definition

Two dg-(commutative) algebras A, B are weakly homotopy equivalent if they can be linked via a zig-zag of algebras where all the maps are cohomology equivalences.

$$A \xrightarrow{\sim} X \xleftarrow{\sim} \ldots \xrightarrow{\sim} Y \xleftarrow{\sim} B$$

- Work of Quillen/Sullivan/Mandell says that the homotopy type $C^*(X, R)$ as an algebra[2] is[3] a complete (weak) homotopy invariant of X.

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- Any homotopy invariant of such algebras will be a homotopy invariant of spaces.
- Want to find such invariants.

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Classical Massey products

Massey triple products

Question (Formality)

When is a dg-algebra equivalent, as an algebra, to its cohomology?

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Answer: Obstructions are given by Massey products.

Definition

Let A be a dg-algebra. Let $a, b, c \in H^\bullet(A)$ by such that $ab = 0$ and $bc = 0$. Let x, y, z be cocycles representing a, b, c and suppose $du = xy$ and $dv = yz$. Then $uz - xv$ is a cocycle and represents a well-defined class of

$$\frac{H^{|a|+|b|+|c|-1}(A)}{aH^{|b|+|c|-1}(A) + H^{|a|+|b|-1}(A)c}$$

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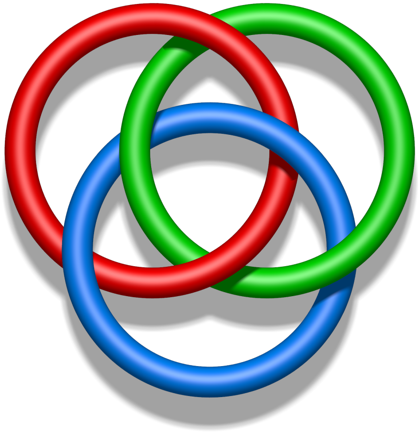
Theorem (Massey [7], 1958)

Weakly equivalent associative algebras have the same Massey products.

The geometric picture



Source: Jim.belk; Wikipedia

Higher Massey products

- The Massey product we gave is a cocycle because A satisfies the associative relation. Algebras that satisfy other relations have Massey products corresponding to those relations. (Muro [9], 2023)
- There are higher order Massey products, defined only when lower order Massey product vanishes. *Spoiler: These will correspond to the syzgyies (relations between relations) of the associative relation.*
- These are defined via a very concrete formula

$$\sum_{0 \leq i < k < j \leq n} (-1)^{|b_{ik}|+1} b_{ik} b_{kn}$$

where $d(b_{ik})$ is a (vanishing) Massey product of degree $k - i$ (Massey, 1958)

Other Massey products

- Similar invariants were defined for dg-Lie algebras (Alladay [1] - Retah [10], 1977)
- There are *matric Massey products* which correspond to more complicated expressions (like $ab + cd + ef = 0$ for example) in the cohomology ring. (May [8], 1968)
- These can be packaged together as the differentials in the *Eilenberg-Moore spectral sequence* which converges to $\mathrm{Tor}^{A}(\mathbb{k}, \mathbb{k})$ from an E^2-page of $\mathrm{Tor}^{H(A)}(\mathbb{k}, \mathbb{k})$.

Coherent vanishing

It is not enough for this spectral sequence to collapse on E_2-page ie. for all Massey products to vanish.
Formality turns out to be them vanishing in a *coherent* way.

Theorem (Deligne, Griffiths, Morgan, Sullivan [4])

Let A be a commutative dg-algebra over $\mathbb{Q}$. Let $\mathfrak{m} = (\mathrm{Sym}(\bigoplus_{i=0}^{\infty} V_i), d)$ be the minimal model for A. Then A is formal if and only if, there is in each V_i a complement B_i to the cocycles Z_i, $V_i = Z_i \oplus B_i$, such that any closed form, a, in the ideal, $I((\bigoplus_{i=0}^{\infty} B_i)$, is exact.

Slogan: Massey products are the local obstructions to commutativity.

Interlude: The world's briefest introduction to algebraic operads

Operads

- An operad is a bit like a group: it abstracts away the properties of some kind of a multiplicative structure like an associative or Lie algebra.

[4] Actually a representation of $\mathbb{S}_n$

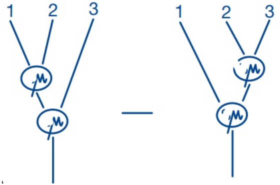
Operads

- An operad is a bit like a group: it abstracts away the properties of some kind of a multiplicative structure like an associative or Lie algebra.
- Algebras are a bit like group representations, they are concrete examples of objects equipped with such a multiplication.
- An operad consists of:
  - A vector space[4] $\mathcal{P}(n)$ for each $n \in \mathbb{N}$. This is the space of all operations that take n-inputs. For the associative operad this has dimension $(n!)$, for the commutative just 1.
  - A composition rule for operations. Given $\mu \in \mathcal{P}(n)$ and $\tau \in \mathcal{P}(m)$, one has rule assigning an operation $\mu \circ_i \tau \in \mathcal{P}(n + m - 1)$
- You can apply an operad to a vector space and then you get the free $\mathcal{P}$-algebra on that space. (Operads are essentially a decomposition of the free $\mathcal{P}$-algebra monad)

[4] Actually a representation of $\mathbb{S}_n$

Operads and trees

- The associative operad is generated by a single arity two operation $\mu = - \cdot - \in \mathcal{P}(2)$
- The free operad $\mathcal{F}(R)$ is made up of sums of trees.
- To get the associative operad we quotient $\mathcal{F}(\mu)$ by an operadic ideal generated by the following element.



- The associative operad is $\mathcal{F}(R)/(E)$, it is *quadratic.*

Operadic Koszul duality

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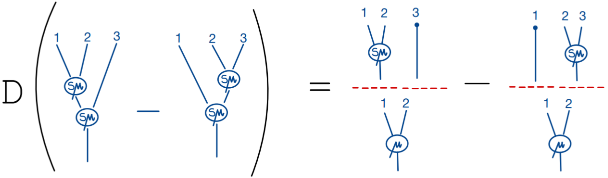
$$(\mathcal{F}(R)/(E))^{\text{i}} = \mathcal{C}(sR, s^2E) \hookrightarrow \mathcal{F}^c(sR)$$

So relations can also all be represented as trees!

Massey products for algebras over algebraic operads

Generalising Massey products

- The idea is that $\mathcal{P}$-Massey products should correspond precisely to co-operations (represented as trees) in the Koszul dual cooperad $\mathcal{P}^{\text{i}}$. The order of the operation corresponds to the weight of the tree.
- You have an inductive map on the weight of the tree given by pruning all the branches at the root of trees.



- The weight zero operation correspond to the initial inputs.

Theorem (FC-Moreno-Fernandez, 2023)

Weakly equivalent $\mathcal{P}$-algebras have the same $\mathcal{P}$-Massey products.

Examples

Specializing to various cases and operads, we recover:

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- Dual numbers operad $\mathcal{D}$: Algebras over $\mathcal{D}$ are bicomplexes. The Massey products are precisely the differentials in the associated spectral sequence.
- Poisson operad: No closed formulae. Here's third order operation.

$$\begin{aligned}[y_{\wedge,(1,2)}, y_{\wedge,(3,4)}] + [z_1, x_{b,(2,3,4)}] + [x_{b,(1,3,4)}, z_2] + 2[z_1, x_{a,(3,4,2)}] \\ +2[y_{[-],(1,3)}, y_{\wedge,(4,2)}] + 2[z_1, x_{a,(2,3,4)}] + 2[x_{a,(1,2,3)}, z_4] + 2[z_3, x_{a,(1,4,2)}] \\ +2[x_{a,(1,4,3)}, z_2] + [x_{b,(2,3,1)}, z_4] + [x_{b,(3,1,2)}, z_4]\end{aligned}$$

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- Example: secondary stable Steenrod operations (computed by Baues [2])
- One may use these to study (strictly) commutative algebras [5]. These are hard to study using more classical tools.

Applications: Producing counterexamples

Cotriple products can be used to produce examples of:

- Commutative algebras A, B over $\mathbb{Z}$ such that $A \otimes \mathbb{Q}$ and $B \otimes \mathbb{Q}$ are weakly equivalent, but $A \otimes \mathbb{F}_p$ and $B \otimes \mathbb{F}_p$ are not.
- Commutative algebras which have a divided power structure on cohomology but which are not weakly equivalent to a divided power algebra.
- Commutative algebras A, B over $\mathbb{F}_p$, which are weakly equivalent as associative algebras but not commutative algebras. This answers a question raised in a recent paper[5].
- (On-going) Commutative algebras A, B over $\mathbb{F}_p$ that are weakly equivalent as E_∞-algebras but not commutative algebras.

[5] R. Campos et al. *Lie, associative and commutative quasi-isomorphism*. To appear in Acta Mathematica. arXiv: 1904.03585 [math.RA].

Rectification

- One has a notion of an E_∞-algebra. This is an algebra that is not commutative 'on the nose' but is up to coherent homotopy.
- A natural question is: when such an E_∞-algebra is weakly equivalent to a commutative algebra?
- There are obstructions: A subset of the cotriple operations, called *higher Steenrod operations*.

Theorem (FC [5])

An E_∞-algebra is rectifiable if and only if its higher Steenrod operations vanish coherently.

Further questions

- Massey products are *effective*. Write a Python program that can effectively compute them.
- Applications in differential geometry. Find a Poisson manifold, that is formal as a manifold (for example, Kähler) and as a Lie algebra but not as a Poisson manifold.
- Operadic Koszul duality is not the only form of Koszul duality. Find Massey products in this context.
- There are *vastly* more complicated generalisations of operads called modular operads and properads. The difference is that these are defined in terms of more general graphs rather than trees. Define Massey products in this context. In particular, try to say something interesting about Calabi-Yau manifolds.

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