The geometry of iterated suspensions

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Introduction

In 1972, JP May published *The geometry of iterated loop spaces* [1], introducing the notion of an operad for the first time. He used this to prove the *recognition principle* which characterizes n-fold loop spaces as algebras over the little n-discs operad. Eckmann-Hilton duality is a (not always valid) heuristic that asserts some notions in algebraic topology are naturally dual to each other. For example; wedge products $\sim$ smash products; suspensions $\sim$ loop spaces. Eckmann-Hilton duality would therefore lead one to suspect that iterated suspensions are coalgebras over the little discs operad. This turns out to be true, but May's proof cannot be dualized. Therefore some new ideas are needed!

Coalgebras in topological spaces

There is no natural notion of a cooperad in topological spaces. Therefore we need to study coalgebras over operads. Let X be a pointed space. The coendomorphism operad has arity n component

$$\mathsf{CoEnd}(X)(n) = \mathrm{Map}(X, X^{\vee n})$$

where the symmetric group permutes the factors in the wedge sum and the composition map is the following

$$\mathsf{CoEnd}(X)(n) \times \mathsf{CoEnd}(X)(i_1) \times \cdots \times \mathsf{CoEnd}(X)(i_n) \to \mathsf{CoEnd}(X)(i_1 + \cdots i_n)$$

$$(f; f_1, \ldots, f_n) \mapsto f \circ (f_1 \vee \cdots \vee f_n)$$

A $\mathcal{P}$-coalgebra is a map

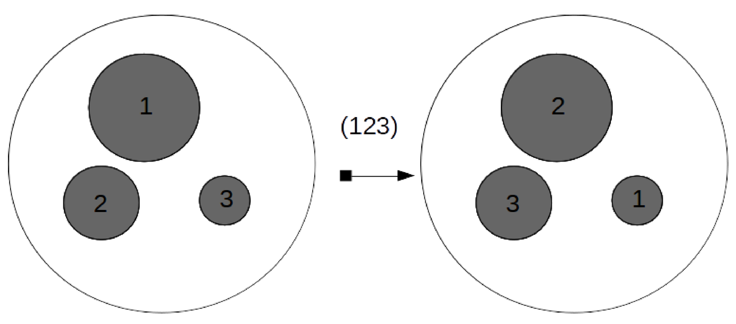
$$\mathcal{P} \to \mathsf{CoEnd}(X)$$

Another way of saying this is that it is a coherent collection of cooperations

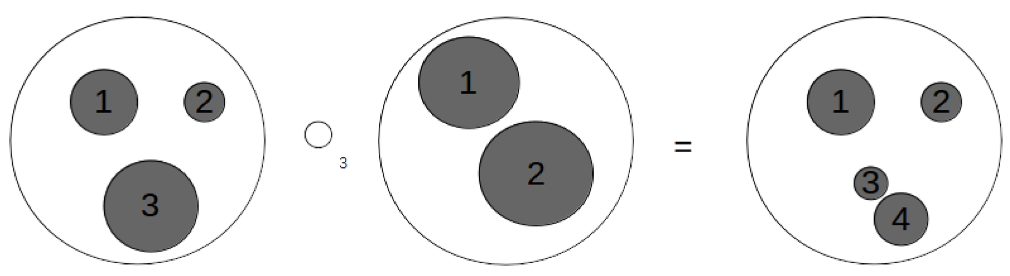
$$\triangle_n : \mathcal{P}(n) \times X \to X^{\vee n}$$

The little discs operad

- The little n-discs space $\mathbb{D}_n(r)$ consists of all pairwise disjoint embeddings of r (labelled) little discs inside a fixed unit n-disc.
- It is a subset of $\mathrm{Map}_{Top}(\sqcup_{i=1}^r D^n, D^n)$, and so equipped with the compact-open topology.
- The symmetric group acts by permuting the discs



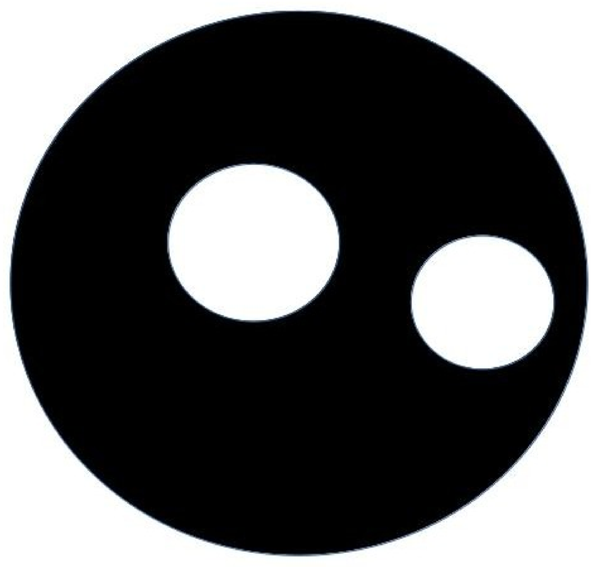
- Operadic composition is via the illustrated substituting procedure.



Theorem 1 (May's recognition principle)**.** *Every n-fold loop space is a $\mathbb{D}_n$-algebra. Conversely, if X is a connected space that is an algebra over $\mathbb{D}_n$ then it has the weak homotopy type of the n-fold loop space of some connected pointed space Y.*

Coalgebra structure on spheres

- We just need to define maps $\triangle_r : \mathbb{D}_n(r) \times S^n \to (S^n)^{\vee r}$.
- Fix some $\theta \in \mathbb{D}_n(r)$. Intuitively, just collapse the black area in the following diagram to get the wedge of two copies of S^2.



Proposition 1. *Every n-fold suspension is a $\mathbb{D}_n$-coalgebra.*

Cofree comonads

- Let $\mathcal{P}$ be a unitary operad and X is a pointed space, we denote

$$\mathrm{Tot}\,(\mathcal{P}, X) := \prod_{n \geq 0} \mathrm{Map}_{S_n}(\mathcal{P}(n), X^{\vee n}).$$

Define the endofunctor in pointed spaces

$$C_\mathcal{P} : \mathsf{Top}_* \to \mathsf{Top}_*$$
$$X \mapsto C_\mathcal{P}(X),$$

where

$$C_\mathcal{P}(X) = \{\alpha = (f_1, f_2, \ldots) \in \mathrm{Tot}\,(\mathcal{P}, X) \mid \pi_i f_n = f_{n-1} d_i \text{ for all } n \geq 2 \text{ and } 1 \leq i \leq n$$

where

$$\pi_i : X^{\vee n} \to X^{\vee n-1}$$

is the map that collapses the i^{th} factor in the wedge sum and

$$d_i : \mathcal{P}(n) \to \mathcal{P}(n-1)$$

is the map given by partially composing with the unit (so for the little discs operad it forgets the disc labelled i).
- *Surprising fact*: The sequence $(f_1, f_2, \ldots)$ always turns out to be determined by f_1. This doesn't usually happen for cofree comonads in other contexts (for example: coassociative coalgebras in dg vector spaces).
- This allows us to define the comonadic structure by manipulating the f_1 and checking what we get is still in the comonad.

Proposition 2. *There is an equivalence of categories between $\mathcal{P}$-coalgebras defined using coendomorphism operad and coalgebras over the comonad $C_\mathcal{P}$.*

Warning: Weakly equivalent operads do not rise to weakly equivalent comonads. For example: $C_{\mathrm{Ass}} = *$ so there are no strictly coassociative coalgebras in spaces.

The little discs operad revisted

The comonad associated to the little dics operad has a further nice property. Its corelations all lie in arity 2. More precisely, a map $f : \mathbb{D}_n(1) \to X$ belongs to $C_{\mathbb{D}_n}(X)$ if, and only if, for all $c_1, c_2 \in \mathcal{C}_n(1)$ little n-discs such that $\mathring{c}_1 \cap \mathring{c}_2 = \emptyset$, then $f(c_1) = *$ or $f(c_2) = *$.

The results

The last observation allows one to construct a concrete homotopy retract proving the following result

Theorem 2 (Coapproximation Theorem). *For every $n \geq 1$, there is a natural morphism of comonads*

$$\alpha_n : \Sigma^n \Omega^n \longrightarrow C_{\mathbb{D}_n}.$$

Furthermore, for every pointed space X, there is an explicit natural homotopy retract of pointed spaces

$$\Sigma^n\Omega^n X \rightleftarrows C_{\mathbb{D}_n}(X) \circlearrowleft$$

In particular, $\alpha_n(X)$ is a weak equivalence.

It is not hard to deduce a recognition principle for the $\Sigma^n\Omega^n$ comonad from the fact that the suspension functor happens preserves equalizers (which is surprising given that it is a left adjoint). Alternatively, Blomquist and Harper have a slightly weaker result with extra connectivity assumptions using May-style arguments with the categorical cobar construction.

Proposition 3. *Let X be a $\Sigma^n\Omega^n$-coalgebra. Then X is naturally isomorphic to the n-fold reduced suspension of a space $P_n(X)$ which can be computed as the equalizer of the following pair of maps:*

$$\Omega^n X \underset{\eta_{\Omega^n X}}{\overset{\Omega^n\gamma}{\rightrightarrows}} \Omega^n\Sigma^n\Omega^n X.$$

Here, η is the unit of the (Σ^n, Ω^n) adjunction, and γ is the $\Sigma^n\Omega^n$-coalgebra structure map of X.

Finally, we can combine both of the previous results with the help of a little abstract nonsense to deduce

Theorem 3 (Corecognition Principle)**.** *Let X be a $\mathbb{D}_n$-coalgebra. Then there is a pointed space $\Gamma^n(X)$, naturally associated to X, together with a weak equivalence of $\mathbb{D}_n$-coalgebras*

$$\Sigma^n\Gamma^n(X) \xrightarrow{\simeq} X,$$

which is a retract in the category of pointed spaces. Therefore, every $\mathbb{D}_n$-coalgebra has the homotopy type of an n-fold reduced suspension and every n-fold reduced suspension has the structure of $\mathbb{D}_n$-coalgebra.

Further applications

The results above hold in the category of topological spaces. We would like to transfer them, and the notion of desuspension, to other more algebraic categories.

- There **may** be a theory of homotopy operations, which is Eckmann-Hilton dual to the work of Cohen-Lada-May in *The Homology of Iterated Loop Spaces* [2]
- One can use an induced coalgebra structure to detect suspensions and recover the fundamental group of a suspension in the category of E_3-chain algebras equipped with *double cobar equivalence*
- More unrelatedly, the little n-discs operad induces some cooperations on the higher Hochschild homology that admit nice descriptions via a HKR-type theorem.

References

[1] J. P. May. *The geometry of iterated loop spaces.* Lecture Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972.

[2] Frederick R. Cohen, Thomas J. Lada, and J. Peter May. *The homology of iterated loop spaces.* Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[3] Oisín Flynn-Connolly, José' M. Moreno-Fernández, and Felix Wierstra. A recognition principle for iterated suspensions as coalgebras over the little cubes operad, 2023.

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