The homotopy theory of the little *n*-discs operad

Oisín Flynn-Connolly

Université Paris-Sud

July 17, 2020

Introduction

- The little n-discs operad $\mathbb{D}_n$ was introduced by J. P. May in 1979 to model algebra structures on n-fold loop spaces.
- We have inclusions

$$\mathbb{D}_1 \hookrightarrow \mathbb{D}_2 \hookrightarrow \cdots \hookrightarrow \mathbb{D}_\infty$$

  Each defines a class of homotopy associative algebras that get steadily 'more commutative'

Why do we care?

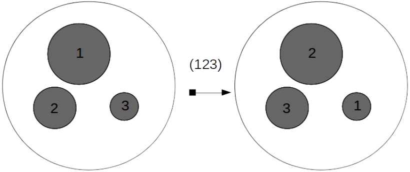
- Provides a 'recognition principle' for n-fold loop spaces.
- The homology of $\mathbb{D}_n$ can be shown to be $Pois^n$. Implies the existence of a Browder bracket of degree $1 - n$ on the homology of n-fold loop spaces.

Plan

1. Introduce the little discs operad and coalgebras over it.
2. Introduce the homotopy theory of operads.
3. Talk about the Barratt-Eccles E_n-operad.
4. Discuss simplicial models for the coendomorphism operad.
5. Look at E_n-algebras in simplicial sets.
6. Look at the homotopy Barratt-Eccles operad (E_1-case)
7. Conjectures and further work.

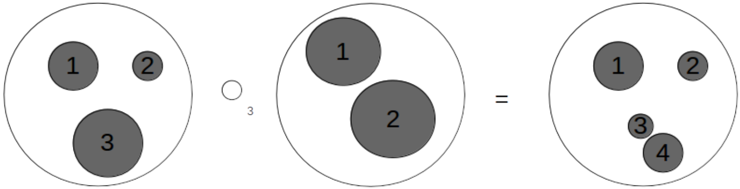
The little n-disc operad: definition

- The little n-discs space $\mathbb{D}_n(r)$ consists of all pairwise disjoint embeddings of r (labelled) little discs inside a fixed unit n-disc.
- It is a subset of $\mathrm{Map}_{Top}(\sqcup_{i=1}^{r} D^n, D^n)$, and so equipped with the compact-open topology.



The little n-disc operad: operadic composition

Operadic composition is via the illustrated substituting procedure.



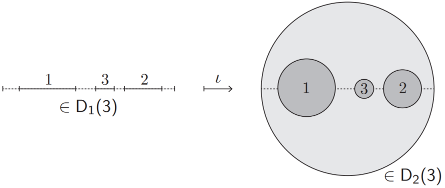
As promised

Theorem (May's recognition principle)

Every n-fold loop space is a $\mathbb{D}_n$-algebra. Conversely, if X is a connected space that is an algebra over $\mathbb{D}_n$ then it has the weak homotopy type of the n-fold loop space of some connected pointed space Y.

The nested sequence of little discs operads

- There is a topological morphism $\mathbb{D}_n(r) \hookrightarrow \mathbb{D}_{n+1}(r)$ given by the equatorial map.
- This sends any little n-disc with centre $\vec{P}$ and radius R to the $(n+1)$-disc with centre $(\vec{P}, 0)$ and radius R.
- For example; see the diagram (image credit to Benoit Fresse).



- This extends to a morphism of operads and defines an infinite sequence of inclusions

$$\mathbb{D}_1 \hookrightarrow \mathbb{D}_2 \hookrightarrow \cdots \mathbb{D}_i \hookrightarrow \mathbb{D}_{i+1} \hookrightarrow \cdots$$

- Define $\mathbb{D}_\infty = \operatorname{colim}_n \mathbb{D}_n$

Eckmann-Hilton duality

- This is a heuristic that asserts some notions in algebraic topology are naturally dual to each other.
- For example; fibration $\sim$ cofibrations; homotopy groups $\sim$ cohomology groups; wedge products $\sim$ smash products; suspensions $\sim$ loop spaces.
- Eckmann-Hilton duality asserts that if all concepts in a theorem are replaced by their dual the theorem should remain true.
- **Dual theorems do not necessarily admit dual proofs.**

Coalgebras of the little n-discs operad

- We would like to study suspensions using the little n-discs operad.
- Problem: No natural notion of cooperad in Top .
- Solution (Moreno-Fernández, Wierstra): Define $\mathscr{P}$-coalgebras as operadic morphisms $\phi : \mathscr{P} \to \mathrm{CoEnd}(X)$ where $\mathrm{CoEnd}(X)$ is the coendomorphism operad of X.

Definition (Coendomorphism operad)

Let X be a **pointed** space. The *coendomorphism operad* $\mathrm{CoEnd}(X)$ has arity r component

$$\mathrm{CoEnd}(X)(r) := \mathrm{Map}_*(X, X^{\vee r})$$

For $r = 0$, set $\mathrm{CoEnd}(X)(0) = \mathrm{Map}_*(X, *) = *$. The symmetric group action permutes the wedge factors in the output. The operadic composition maps are defined by

$$\gamma(f, f_1, \cdots, f_n) := (f_1 \vee \cdots \vee f_n) \circ f$$

Coalgebras of the little n discs operad

Example (The pointed n-sphere)

- Suffices to define maps $\triangle_r : \mathbb{D}_n(r) \times S^n \to (S^n)^{\vee r}$.
- Let $x = (f_1, f_2, \ldots, f_r) \in \mathbb{D}_n(r)$ (note: $f_i : D^n \to D^n$) and $y \in D^n$. Define $\triangle'_r : \mathbb{D}_n(r) \times D^n \to (\bigsqcup_{i=1}^r D^n) \cup \{*\}$

$$\triangle'_r(x, y) = \begin{cases} * & \text{if } y \notin f_i(D^n) \forall i \\ f_i^{-1}(y) & \text{otherwise.} \end{cases}$$

- $\triangle'_r$ descends to a map $\mathbb{D}_n(r) \times S^n \to (S^n)^{\vee r}$, by collapsing everything the boundary of every disc.

Theorem

The n-fold suspensions are coalgebras over $\mathbb{D}_n$.

The model category of operads

- Let **C** be a closed symmetric monoidal model category. Is there an induced structure on operads over it?

Theorem (Berger-Moerdjik, 2003)

Let $(\mathbf{C}, \otimes, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a closed symmetric monoidal cofibrantly generated model category with unit I such that

1. *I is cofibrant*
2. *the over-category $\mathbf{C}/I$ has a symmetric monoidal fibrant replacement functor*
3. ***C** admits a commutative Hopf interval.*

Then there is a cofibrantly generated model structure on the category of reduced operads, in which a morphism of reduced operads $f : \mathscr{P} \to \mathscr{Q}$ is a fibration (resp. weak equivalence) if and only if the induced map $\mathscr{P}(n) \to \mathscr{Q}(n)$ is a fibration (resp. weak equivalence) in the category $\mathbf{C}^{\mathbb{S}_n}$ for all $n \in \mathbb{N}_0$.

The simplicial operad Γ

- Motivation: We want a model for the $\mathbb{D}_\infty$ operad.
- Each element of $\Gamma(r)_n$ is a $n+1$-tuple $\langle \sigma_0, \dots \sigma_n \rangle$ of elements from $\mathbb{S}_r$.
- Face and degeneracy maps given by repeating or omitting elements of the tuple. (In other words, this is universal bundle construction applied to $\mathbb{S}_r$)
- The symmetric group action is componentwise within each tuple.
- Composition is also componentwise. Within each component, the composition is that of the associative operad.
- For example; let $(1,2), id \in \mathbb{S}_2$

$$
\begin{aligned}
\langle (1,2), id \rangle \circ_2 \langle (1,2), id \rangle &= \langle (1,2) \circ_2 (1,2), id \circ_2 id \rangle \\
&= \langle (1,3,2), id \rangle \in \mathbb{S}_3 \times \mathbb{S}_3
\end{aligned}
$$

The Barratt-Eccles E_k-operad

- Defined via the *Smith filtration* of the symmetric operad.
- Let τ be a permutation in $\mathbb{S}_r$. For $i < j$, let $\tau|_{i,j}$ be 0 if $\tau(i) < \tau(j)$ and 1 otherwise.
- Consider a simplex $\sigma = (\sigma_0, \ldots, \sigma_n)$ in $\Gamma(r)_n$. Let μ_{ij}^{σ} be the number of times the sequence $(\sigma_0|_{i,j}, \ldots, \sigma_n|_{i,j})$ changes values.
- The Barratt-Eccles E_k operad $\Gamma^{(k)}(r)_n$ consists of the $\sigma \in \Gamma(r)_n$ such that $\mu_{ij}^{\sigma} < k$ for all $i < j$.
- One obtains a sequence of inclusions

$$\Gamma^{(1)} \hookrightarrow \Gamma^{(2)} \hookrightarrow \ldots \hookrightarrow \Gamma^{(i)} \hookrightarrow \ldots \hookrightarrow \Gamma$$

Equivalence with the little discs operad (sketch)

Theorem

The geometric realization of the Barratt-Eccles E_n operad is weakly equivalent to the little n-discs operad.

- One can decompose the simplicial operad into contractible cells labelled by a certain operadic poset $\mathcal{K}$, in a way compatible with the operad structure.
- Note: This dissection is compatible with the Smith filtration.
- One shows that the little n-discs operad can also be dissected into contractible cells labelled by $\mathcal{K}$.
- Geometrically realize the simplicial operad.
- Both $|\Gamma^{(n)}|$ and $\mathbb{D}_n$ admit retraction onto the realisation of the nerve of the poset and so are weakly equivalent.

Simplicial coendomorphism operads: Part 1

- Want to study the analogue of the the coendomorphism operad for simplicial sets.
- The naïve notion is to

$$\mathrm{Map}_{\mathsf{Set}_\triangle}(X, X^{\vee r}).$$

- This is not great - consider $S^1 = \triangle^1/\partial\triangle^1$.
- In fact it doesn't even have the right homotopy type.

Simplicial coendomorphism operads: Part 2

There is a natural way to transfer operads into simplicial sets.

Definition

Let $\mathscr{P}$ be a topological operad. We define an operad $S_\bullet\mathscr{P}$ over $\mathrm{Set}_\triangle$ with arity n component

$$(S_\bullet\mathcal{P})(n) := S_\bullet(\mathcal{P}(n))$$

where $S_\bullet$ is the singular chains functor. The action of $\sigma \in \mathbb{S}_n$ on $S_\bullet\mathcal{P}(n)$ is given by $S_\bullet\mathcal{P}(n) * \sigma := S_\bullet(\mathcal{P}(n) * \sigma)$. The operadic composition map is $\gamma_{S_\bullet\mathcal{P}} := S_\bullet(\gamma_\mathcal{P})$ and we take the unit to be the simplex $[\triangle^0 \to 1_{\mathbf{Top}'}] \in S_\bullet\mathcal{P}(1)$.

Simplicial operads: Part 2

- This tells us what the 'correct' simplicial coendomorphism operad should look like (up to homotopy type).

$$S_\bullet(\mathrm{CoEnd}_{\mathrm{Top}}(X))$$

- If X is an n-fold simpicial suspension this has the stucture of homotopy Barratt-Eccles E_n-coalgebra via the morphism

$$S(\Phi) : S_\bullet(\mathbb{D}_n) \to S_\bullet(\mathrm{CoEnd}_{\mathrm{Top}}(|X|))$$

The simplicial coendomorphism operad: Part 3

Let X be a finite simplicial set.

- Recall Kan's Ex^∞-functor is defined as the colimit of the following diagram

$$X \to \mathrm{Ex}(X) \to \mathrm{Ex}^2(X) \to \cdots \mathrm{Ex}^i(X) \to \cdots$$

  where Ex is the right adjoint to the barycentric subdivision functor sd .
- Consider the operad

$$\mathrm{CoEnd}_{\mathrm{Set}_\triangle}(X)(r) = \mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(X^{\vee r}))$$

- Notice that every morphism $f : X \times \triangle^m \to \mathrm{Ex}^\infty(X^{\vee r})$ factors through $\mathrm{Ex}^{N_f}(X^{\vee r})$ for some finite N_f.
- We therefore have an adjoint map

$$(f, N_f) : \mathrm{sd}^{N_f}(X \times \triangle^m) \to X^{\vee r}.$$

The simplicial coendomorphism operad: Part 3

Let $f \in \mathrm{CoEnd}(X)(r)_m$ and $f_i \in \mathrm{CoEnd}(X)(n_i)_m$ for $1 \leq i \leq r$. We define their composition as the adjoint $\overline{F} : X \times \triangle^m \to X^{\vee n_1 + \cdots n_r}$ to the map $F : \mathrm{sd}^{N+N_f}(X \times \triangle^m) \to X^{\vee n_1+\cdots n_r}$

$$F : \mathrm{sd}^{N+N_f}(X \times \triangle^m) \xrightarrow{\mathrm{sd}^N(\delta_{X\times\triangle^m})} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m))$$
$$\xrightarrow{\mathrm{sd}^N(\mathrm{id}\times \mathrm{sd}^{N_f}(\pi_2))} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\mathrm{sd}^{N_f}(\triangle^m)) \xrightarrow{a} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\triangle^m)$$
$$\xrightarrow{\mathrm{sd}^N((f,N_f))} \mathrm{sd}^N(X^{\vee r}\times\triangle^m) \xrightarrow{b} \mathrm{sd}^N(X\times\triangle^m)^{\vee r} \xrightarrow{(f_1,N)\vee\cdots\vee(f_r,N)} X^{\vee n_1+\cdots n_r}$$

- N is the integer $\max(N_{f_1}, \ldots, N_{f_n})$.
- $(f, N) : \mathrm{sd}^N(X \times \triangle^m) \to X^{\vee n_i}$ is the last vertex map.
- $\delta_{\mathrm{sd}^{N_f}(X\times\triangle^m)} : \mathrm{sd}^{N_f}(X \times \triangle^m) \to \mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m)$ is the diagonal map.
- $\pi_2 : X \times \triangle^m \to \triangle^m$ is the projection.

E_n-coalgebras in simplicial sets

Definition

Let $\mathscr{P}$ be an operad in simplicial sets. We shall say that a finite simplicial set X is a *$\mathscr{P}$-coalgebra* if there exists an operadic morphism $\Phi : \mathscr{P} \to \mathrm{CoEnd}(X)$.

- *One can show* that $\mathrm{CoEnd}_{\mathsf{Set}_\triangle}(X)$ is weakly equivalent to $S_\bullet \mathrm{CoEnd}_{\mathsf{Top}}(X)$ and it follows from the homotopy transfer principle that there is a homotopy Barratt-Eccles structure on simplicial suspensions.
- The next natural question is to describe it.
- Therefore we need a model for the homotopy Barratt-Eccles operad.

The homotopy Barratt-Eccles operad

- Need a cofibrant model for the Barratt-Eccles operad.
- We apply the Boardman-Vogt resolution, a canonical cofibrant replacement functor for the operadic model category.

Theorem

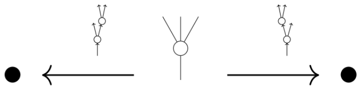
Let $\mathcal{G}$ be the directed graph on

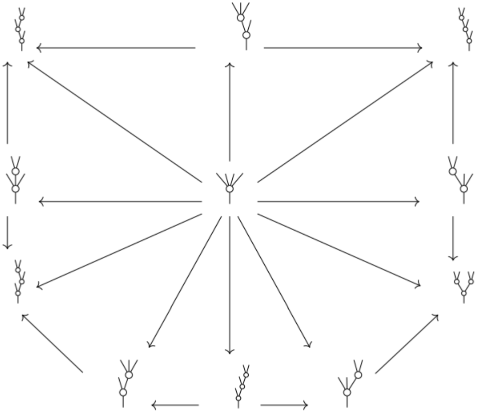
$$\frac{3P_{n-1}(3) - P_{n-2}(3)}{4n}$$

vertices, where P_n is the n^{th} Legendre polynomial and where each vertex v_T is labelled by an n-ary non-unital tree with no vertices of arity 0. The edges of $\mathcal{G}$ are defined as follows; there is a directed edge from v_T to v_S iff one can obtain T from S by collapsing internal edges. Then this graph forms a poset $\overline{\mathcal{G}}$ and

$$W(\triangle^1, E_1)(n) = \coprod_{\mathbb{S}_n} \mathcal{N}(\overline{\mathcal{G}})$$

The W-construction of the E_1 operad





Conjecture

- To describe the coalgebra structure, we start with the simplest case, the suspension of the point (which is S^1) and the Barratt-Eccles E_1-operad.
- The set $\mathrm{Ex}^\infty(S^1)_1$ consists of eventually constant strings of the form

$$\bullet \xrightarrow{\sigma} \bullet \xleftarrow{*} \bullet \xrightarrow{\sigma} \bullet \xleftarrow{*} \bullet \xrightarrow{*} \bullet \xleftarrow{*} \bullet \xrightarrow{\sigma} \cdots$$

  where $*$ is the degeneracy of the base point and σ is the nondegenerate 1-simplex.
- The simplicial set $W(\triangle^1, \mathrm{Assoc})(2)$ consists of two disjoint 1-simplices p, q. We conjecture that the coalgebra structure is given in arity 2

$$W(\triangle^1, \mathrm{Assoc})(2) \times S^1 \to \mathrm{Ex}^\infty(S^1 \vee S^1)$$

$$p \mapsto (\bullet \xrightarrow{\alpha} \bullet \xleftarrow{*} \bullet \xrightarrow{\beta} \bullet \xleftarrow{*} \bullet \xrightarrow{*} \bullet \xleftarrow{*} \bullet \xrightarrow{*} \cdots)$$

$$q \mapsto (\bullet \xrightarrow{*} \bullet \xleftarrow{\alpha} \bullet \xrightarrow{*} \bullet \xleftarrow{\beta} \bullet \xrightarrow{*} \bullet \xleftarrow{*} \bullet \xrightarrow{*} \cdots)$$

Further work

We suggest that it may possible to

- Prove a recognition principle for iterated suspensions
- Construct analogues of the Dyer-Lashof and Kudo-Araki operations for iterated suspensions.

Acknowledgements

I'd like to thank my supervisors Felix Wierstra and Grégory Ginot. Thanks for listening, any questions?