The Category of Quasi-Parabolic Coherent Sheaves

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1 Introduction

A (quasi-)parabolic vector bundle (over an algebraic curve X) with lengths $\{w_p\}_{p\in X}$ is a vector bundle with equipped with a filtration of the fibre E_p:

$$E_p \supset E_p^1 \supset E_p^2 \cdots \supset E_p^{w_p}$$

for each $p \in X$, with $w_p = 0$ for all but finitely many p.
The category of parabolic vector bundles over X is not abelian in the following sense (but it is quasi-abelian as will be explained later). An additive category is said to be *abelian* if all kernels and cokernels exist, and images are isomorphic to coimages.
We can construct an abelian envelope of the category of parabolic vector bundle $\mathrm{PVec}_w(X)$ in two different ways. Firstly, we can extend the concept of parabolic vector bundles to realm of coherent sheaves. A (quasi-) parabolic coherent sheaf F with lengths $\{w_p\}_{p\in X}$ is, for each $p \in X$, a sequence of sheaves [1]

$$F^{(p,0)} \to F^{(p,1)} \to F^{(p,2)} \to \cdots \to F^{(p,w_p-1)} \to F^{(p,0)} \otimes O(p)$$

One requires that the composition $F^{(p,i)} \to F^{(p,i+w_p)} = F^{(p,i)} \otimes O(p)$ is canonical for all $0 \leq i < w_p$ and that $\forall p, q \in X$, $F^{(p,0)} = F^{(q,0)}$.
The category $\mathrm{PCoh}_w(X)$ of such objects is abelian, and we can identify locally free parabolic coherent sheaves with parabolic vector bundles.
The other way to find an abelian envelope for $\mathrm{PVec}_w(X)$ is Schneiders' construction of the left heart of a quasi-abelian category.
Our aim was to show that these two approaches yield equivalent categories. This was already known in the case of ordinary coherent sheaves and vector bundles.

2 Preliminaries

Definition 2.1. An additive category $\mathcal{A}$ is called *quasi-abelian* if it has kernels, cokernels, the pushout of a strict monomorphism is a strict monomorphism and the pullback of a strict epimorphism is a strict epimorphism.

Definition 2.2. We say that an ordered pair of full categories $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ is a *torsion pair* if:

- $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
- $\forall X \in A, \exists T \in \mathcal{T}, F \in \mathcal{F}$ such that we have an exact sequence

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

In this case, $\mathcal{T}$ and $\mathcal{F}$ are, respectively, called the torsion and torsion-free components of the torsion pair.

Definition 2.3. A torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ is called *cotilting* if $\forall X \in \mathcal{A}$, there exists $F \in \mathcal{F}$ with an epimorphism from F to X.

Definition 2.4. A (quasi-)parabolic vector bundle (over an algebraic curve X) with lengths $\{w_p\}_{p\in X}$ is a vector bundle with equipped with a filtration of the fibre:

$$E_p \supset E_p^1 \supset E_p^2 \cdots \supset E_p^{w_p}$$

for each $p \in X$. with $w_p = 0$ for all but finitely many p.

Definition 2.5. A (quasi-) parabolic coherent sheaf F with lengths $\{w_p\}_{p\in X}$ is, for each $p \in X$, a sequence of sheaves [1]

$$F^{(p,0)} \to F^{(p,1)} \to F^{(p,2)} \to \cdots \to F^{(p,w_p-1)} \to F^{(p,0)} \otimes O(p)$$

One requires that the composition $F^{(p,i)} \to F^{(p,i+w_p)} = F^{(p,i)} \otimes O(p)$ is canonical for all $0 \le i < w_p$ and that $\forall p, q \in X$, $F^{(p,0)} = F^{(q,0)}$.

Definition 2.6. A *morphism* in the category of parabolic vector bundles is a morphism of vector bundles such that the induced map on fibres carries E_p^i into F_p^i for all $0 \le i \le w_p$.

Definition 2.7. A *morphism* in the category of parabolic coherent sheaves is a collection of morphisms $F^{(p,i)} \to G^{(p,i)}$ such that the resulting diagrams commute.

3 Properties of torsion pairs

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$.

Lemma 3.1. *For all $X \in \mathcal{A}$ the short exact sequence*

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$ is unique up to isomorphism.

Proof. Given $X \in \mathcal{A}$, suppose there are two such sequences. Then

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & T' & & & & \\ & & & & \downarrow f' & & & & \\ 0 & \longrightarrow & T & \xrightarrow{f} & X & \xrightarrow{g} & F & \longrightarrow & 0 \\ & & & & \downarrow g' & & & & \\ & & & & F' & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

with $T, T' \in \mathcal{T}$ and $F, F' \in \mathcal{F}$. Then $T' \to F$ is the zero morphism so $g \circ f' = 0$. So f' factors through the $\ker(g) = f$. Similarily, f factors through f'. So $f' \circ \alpha_1 = f$ where $\alpha_1 : T \to T'$ and $f \circ \alpha_2 = f'$ where $\alpha_2 : T \to T'$. Then $f \circ \alpha_2 \circ \alpha_1 = f$, so $\alpha_1 \circ \alpha_2 = \mathrm{Id}_T$ as f is monomorphic. Similarily, $\alpha_2 \circ \alpha_1 = \mathrm{Id}_{T'}$. So $T \cong T'$. Thus

$$\begin{array}{ccccccccc} T & \longrightarrow & X & \longrightarrow & F & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \phi & & \downarrow \cong & & \downarrow \cong \\ T' & \longrightarrow & X & \longrightarrow & F' & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

$g' \circ f = 0$ which implies g' factors through g. So ϕ exists. Applying the Five Lemma, we see ϕ is an isomorphism. □

Lemma 3.2. *Suppose $X \in \mathcal{A}$. If $Hom(T, X) = 0$ $\forall T \in \mathcal{T}$ then $X \in \mathcal{F}$.*

Proof. We have

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

for $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Therefore $T = 0$ implying $X \cong F \in \mathcal{F}$. □

Lemma 3.3. *$\mathcal{F}$ is closed under subobjects.*

Proof. Suppose $X \hookrightarrow F \in \mathcal{F}$. Then any morphism from $\mathcal{T}$ to X extends to F. Thus $\mathrm{Hom}(T, X) = 0 \; \forall T \in \mathcal{T}$. By Lemma 3.2, $X \in \mathcal{F}$. □

Lemma 3.4. *$\mathcal{F}$ is closed under extensions.*

Proof. Let

$$0 \longrightarrow F_1 \xrightarrow{\alpha} X \xrightarrow{f} F_2 \longrightarrow 0$$

for $F_1, F_2 \in \mathcal{F}$.
There exists $T \in \mathcal{T}$ and $F_3 \in \mathcal{F}$ such that:

$$0 \longrightarrow T \xrightarrow{\beta} X \longrightarrow F_3 \longrightarrow 0$$

Now $T \to F_2$ is the zero morphism so $f \circ \beta = 0$.
So β factors through $\ker(f) = \alpha$. Now $\mathrm{Hom}(T, F_1) = 0$ which implies $\beta = 0$ and thus $X \cong F_3$. □

Lemma 3.5. *$\mathcal{F}$ has cokernels.*

Proof. For a given morphism α in $\mathcal{F}$ denote its cokernel in $\mathcal{A}$ by Q. Then we have:

$$0 \longrightarrow T \longrightarrow Q \longrightarrow F \longrightarrow 0$$

where $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
Verifying that F is $\mathrm{coker}(\alpha)$ in $\mathcal{F}$ is straightforward. □

Lemma 3.6. *The pullback of a strict epimorphism in $\mathcal{F}$ is a strict epimorphism.*

Proof. Let $f : B \to A$ and $g : C \to A$ in $\mathcal{F}$ with f a strict epimorphism. Then f is the cokernel of $\mathrm{Ker}\, f \to B$ in $\mathcal{F}$. Take the pullback P of f and g in $\mathcal{A}$

$$\begin{array}{ccccc} \mathrm{Ker}\,\bar{f} & \xrightarrow{\bar{k}} & P & \xrightarrow{\bar{f}} & C \\ & & \downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle g} \\ \mathrm{Ker}\, f & \xrightarrow{k} & B & \xrightarrow{f} & A \end{array}$$

Consider the kernels (in $\mathcal{A}$), $\mathrm{Ker}\,\bar{f}$ of $\bar{f}$ and $\mathrm{Ker}\, f$ of f. [5] Since $\mathcal{A}$ is abelian we know that $\bar{f}$ is monic and $\mathrm{Ker}\,\bar{f} \cong \mathrm{Ker}\, f$ with $k = \bar{g} \circ \bar{k}$
Since the kernel of f is the same in $\mathcal{A}$ and $\mathcal{F}$, $\mathrm{Ker}\,\bar{f} \cong \mathrm{Ker}\, f \in \mathcal{F}$
Thus the exact sequence:

$$0 \longrightarrow \mathrm{Ker}\,\mathcal{F} \xrightarrow{\bar{k}} P \xrightarrow{\bar{f}} C \longrightarrow 0$$

with $\mathrm{Ker}\,\mathcal{F}$ and C in $\mathcal{F}$ implies $P \in \mathcal{F}$, as $\mathcal{F}$ is closed under extensions. Clearly, $(\bar{f}, \bar{g})$ is the pullback of (f, g) in $\mathcal{F}$, and $\bar{f}$ is the cokernel of $\bar{k}$ in $\mathcal{F}$ and is therefore a strict epimorphism. □

Lemma 3.7. *If f is a strict monomorphism, then the cokernel* $\mathrm{coker}\, f$ *in $\mathcal{A}$* $(\mathrm{Coker}_{\mathcal{A}}\, f)$ *is isomorphic to the cokernel* $\mathrm{coker}\, f$ *in $\mathcal{F}$* $(\mathrm{Coker}_{\mathcal{F}}\, f)$.

Proof.

$$
\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & T & & & & \\
 & & & & & & \downarrow & & & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{c} & \operatorname{Coker}_{\mathcal{A}} f & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\phi} & & \downarrow{\cong} & & \downarrow{\cong} \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \operatorname{Coker}_{\mathcal{F}} f & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & & & & \downarrow & & & & \\
 & & & & & & 0 & & & &
\end{array}
$$

We know f is the kernel of $B \to \operatorname{Coker}_{\mathcal{F}} f$ which is epimorphic as the composition $\phi \circ c$ of epimorphic maps, so the bottom row is exact. Applying the Five Lemma, we see ϕ is an isomorphism. □

Lemma 3.8. *The pushout of a strict monomorphism in $\mathcal{F}$ is a strict monomorphism.*

Proof. Let $f : A \to B$, $g : A \to C$ in $\mathcal{F}$ with f a strict monomorphism in $\mathcal{F}$. Then f is a monomorphism in $\mathcal{A}$ as its kernel is the same as in $\mathcal{F}$. Then f is the kernel of $C \to \operatorname{Coker}_{\mathcal{F}} f$ in $\mathcal{F}$, and $\operatorname{Coker}_{\mathcal{F}} f$ is the cokernel of f in $\mathcal{A}$. Take the pushout P in $\mathcal{A}$ of f and g as in the diagram:

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{c} & \operatorname{Coker} f \\
\downarrow{g} & & \downarrow{\bar{g}} & & \\
C & \xrightarrow{\bar{f}} & P & \xrightarrow{\bar{c}} & \operatorname{Coker} \bar{f}
\end{array}
$$

Consider the cokernel $\operatorname{Coker} \bar{f}$ of $\bar{f}$ in $\mathcal{A}$. As $\mathcal{A}$ is abelian we know $\bar{f}$ is monic and $\operatorname{Coker} \bar{f} \cong \operatorname{Coker} f$ with $c = \bar{c} \circ \bar{g}$. Therefore we have the exact sequence

$$
0 \longrightarrow C \xrightarrow{\bar{f}} P \xrightarrow{\bar{c}} \operatorname{Coker} \bar{f} \longrightarrow 0
$$

with C and $\operatorname{Coker} \bar{f} \cong \operatorname{Coker} f \in \mathcal{F}$. Thus $P \in \mathcal{F}$, and clearly $(\bar{f}, \bar{g})$ is the pushout of (f, g) in $\mathcal{F}$. □

Theorem 3.9. *$\mathcal{F}$ is a quasi-abelian category.*

Proof. That $\mathcal{F}$ is an additive category is trivial. $\mathcal{F}$ has kernels by Lemma 3.3. $\mathcal{F}$ has cokernels by Lemma 3.5. The pushout of a strict monomorphism in $\mathcal{F}$ is a strict monomorphism by Lemma 3.8. The pullback of a strict epimorphism in $\mathcal{F}$ is a strict epimorphism by Lemma 3.6. □

Lemma 3.10. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. There is a functor $t : \mathcal{A} \to \mathcal{T}$ whose object function takes $X \mapsto T$ with $0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$ exact in $\mathcal{A}$, $F \in \mathcal{F}$ and $T \in \mathcal{T}$.*

Proof. Let $t(X) = T$ where $0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$ is the unique (up to isomorphism) exact sequence in $\mathcal{A}$ with $F \in \mathcal{F}$ and $T \in \mathcal{T}$.
Let $\phi : X \to X'$ in $\mathcal{A}$. We have

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T & \xrightarrow{g} & X & \xrightarrow{f} & F & \longrightarrow & 0 \\
 & & \downarrow{t(\phi)} & & \downarrow{\phi} & & & & \\
0 & \longrightarrow & T' & \xrightarrow{g'} & X' & \xrightarrow{f'} & F' & \longrightarrow & 0
\end{array}
$$

$f' \circ \phi \circ g = 0$ implies $\phi \circ g$ factors through g' as g' is the kernel of f'.
Thus $\phi \circ g = g \circ t(\phi)$ and $t(\phi)$ is unique. Uniqueness and the commutativity of:

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & & \downarrow \phi \\ T & \longrightarrow & X' \\ \downarrow & & \downarrow \phi' \\ T'' & \longrightarrow & X'' \end{array}$$

implies $t(\phi' \circ \phi) = t(\phi') \circ t(\phi)$ □

Remark: $t(\cdot)$ is right adjoint to the inclusion functor $\mathcal{F} \to \mathcal{A}$.

4 The left heart construction

We will use without proof the following result of Schneider.

Theorem 4.1. *[2] Let $\mathcal{E}$ be a quasi-abelian category. Let $\mathcal{A}$ be an abelian category with the canonical inclusion $J : \mathcal{E} \to \mathcal{A}$. Then there exists an abelian category $\mathcal{LH}(\mathcal{E})$. Suppose the functor J is fully faithful and that:*

(a) *For any monomorphism*

$$X \to J(F)$$

of $\mathcal{A}$ there is an object F' of $\mathcal{E}$ and an isomorphism

$$X \simeq J(F'),$$

(b) *For any object X of $\mathcal{A}$, there is an epimorphism*

$$J(F) \to X$$

where F is an object of $\mathcal{E}$

Then, J extends to an equivalence of categories:

$$\mathcal{LH}(\mathcal{E}) \approx \mathcal{A}$$

Theorem 4.2. *If $(\mathcal{T}, \mathcal{F})$ is a cotilting torsion pair in an abelian category $\mathcal{A}$ then $\mathcal{A} \approx \mathcal{LH}(\mathcal{F})$*

Proof. We verify the conditions of Theorem 4.1 for $\mathcal{E} = \mathcal{F}$ and J the embedding of $\mathcal{F}$ into $\mathcal{A}$. $\mathcal{F}$ is quasi-abelian by Theorem 3.9. It is obvious that J is fully faithful. By Lemma 3.4, F is closed under extensions, and therefore satisfies condition **(a)**. Condition **(b)** is exactly that $(\mathcal{T}, \mathcal{F})$ is cotilting. □

5 Application to vector bundles

Lemma 5.1. *Let $F \in \mathrm{Coh}(X)$. Then $t(F \otimes O(p)) \cong t(F) \otimes O(p)$*

Proof. Let $S : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$ be the functor $F \mapsto F \otimes O(p)$. Let $S^{-1}(F) = F \otimes O(-p)$. Then $S^{-1}(S(F)) \cong F \cong S(S^{-1}(F))$. Let T be a torsion coherent sheaf. Clearly $T \otimes O(p)$ is also torsion so $S(\mathcal{T}) \subset \mathcal{T}$. Similarly $S^{-1}(\mathcal{T}) \subset \mathcal{T}$ so $S^{-1}(S(\mathcal{T})) = \mathcal{T} \subset S(\mathcal{T})$. Therefore $S(\mathcal{T}) = \mathcal{T}$.
$X \in \mathcal{F} \Leftrightarrow \mathrm{Hom}(T, X) = 0 \ \forall T \in \mathcal{T} \Leftrightarrow \mathrm{Hom}(S(T), S(X)) = 0 \ \forall T \in \mathcal{T}$ by the invertibility of S $\Leftrightarrow$

$\mathrm{Hom}(T, S(X)) = 0 \; \forall T \in \mathcal{T}$ as $S(\mathcal{T}) = \mathcal{T} \Leftrightarrow S(X) \in \mathcal{F}$. Therefore $S(\mathcal{F}) = \mathcal{F}$.
Let $X \in \mathcal{A}$ and consider an exact sequence

$$0 \longrightarrow t(X) \longrightarrow X \longrightarrow F \longrightarrow 0$$

with $t(X) \in \mathcal{T}$ and $F \in \mathcal{F}$
S is an equivalence of categories and therefore exact, giving

$$0 \longrightarrow S(t(X)) \longrightarrow S(X) \longrightarrow S(F) \longrightarrow 0$$

with $S(t(X)) \in \mathcal{T}$ and $S(F) \in \mathcal{F}$.
So the uniqueness of the above sequence implies $S(t(X)) \cong t(S(X))$. □

Theorem 5.2. *The categories* $\mathrm{PTCoh_w}(X)$ *of parabolic torsion coherent sheaves on* X *and* $\mathrm{PVec_w}(X)$ *of parabolic vector bundles on* X *form a torsion pair in* $\mathrm{PCoh_w}(X)$

Proof. Let $T^\bullet \in \mathrm{PTCoh}_w(X)$ and $F \in \mathrm{PVec}_w(X)$. Then $T^{(p,i)} \in \mathrm{TCoh}_w(X)$ and $F^{(p,i)} \in \mathrm{Vec}(X)$ $\forall i \in \{1, 2, \cdots w_p\}$.
Since $\mathrm{Hom}(T^{(p,i)}, F^{(p,i)}) = 0 \; \forall p \in X$ and $0 \leq i \leq w_p$ it is obvious that there are no nontrivial morphisms from $T^\bullet$ to $F^\bullet$. So $\mathrm{Hom}(\mathrm{PTCoh}_w, \mathrm{PVec}_w) = 0$.
Let $(F^{(p,i)})_{p \in X}^{i \leq w_p} \in \mathrm{PCoh}(X)$ Let $T^{(p,i)} = t(F^{(p,i)})$ be the torsion component of $F^{(p,i)}$. For each p this induces a commutative diagram.

$$\begin{array}{ccccccccc}
t(F) & \xrightarrow{t(\phi^{(p,0)})} & t(F^{(p,1)}) & \longrightarrow & \cdots & \longrightarrow & t(F^{(p,n-1)}) & \xrightarrow{t(\phi^{(p,n-1)})} & t(F \otimes O(p)) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
F & \xrightarrow{\phi^{(p,0)}} & F^{(p,1)} & \longrightarrow & \cdots & \longrightarrow & F^{(p,n-1)} & \xrightarrow{\phi^{(p,n-1)}} & F \otimes O(p)
\end{array}$$

As $t(F \otimes O(p)) \cong t(F) \otimes O(p)$ and $t()$ is a functor, the top row of this diagram is a parabolic structure of $t(F)$ at p.
Thus $t(F^\bullet) := T^{(p,i)}$ is a parabolic torsion coherent sheaf. As cokernels are taken componentwise in $\mathrm{PCoh}(X)$

$$F^\bullet / t(F^\bullet) = F^{(p,i)} / t(F^{(p,i)})$$

and by definition of $t(F^{(p,i)})$ we have exact sequences:

$$0 \longrightarrow t(F^{(p,i)}) \longrightarrow F^{(p,i)} \longrightarrow F^{(p,i)}/t(F^{(p,i)}) \longrightarrow 0$$

with $F^{(p,i)}/t(F^{(p,i)}) \in \mathrm{Vec}(X)$, giving the exact sequence:

$$0 \longrightarrow t(F^\bullet) \longrightarrow F^\bullet \longrightarrow F^\bullet/t(F^\bullet) \longrightarrow 0$$

and $t(F^\bullet) \in \mathrm{PTCoh}_w(X), \; F^\bullet/t(F^\bullet) \in \mathrm{PVec}_w(X)$ □

Theorem 5.3. *Let* $(F^{(p,i)})_{p \in S}^{1 \leq i \leq w_p} \in \mathrm{PCoh}(X)$. *Then there exists an epimorphism* $E^\bullet \twoheadrightarrow F^\bullet$ *with* $E^\bullet \in \mathrm{PVec_w}(X)$

Proof. We consider first only $F^\bullet \in \mathrm{PCoh}(X)$ with parabolic structure only at one point p. We know there exists an epimorphism $E_0 \in \mathrm{Vec}(X)$ with $f_0 : E_0 \twoheadrightarrow F^0$. Then we obtain the following diagram

$$\begin{array}{ccccccccc}
E_0 & \xrightarrow{\cong} & E_0 & \xrightarrow{\cong} & \cdots & \xrightarrow{\cong} & E_0 & \xrightarrow{n_E} & E_0 \otimes O(p) \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \phi^0 \circ f_0} & & & & \downarrow & & \downarrow{\scriptstyle f_0 \otimes id_{O(p)}} \\
F^0 & \xrightarrow{\phi^0} & F^1 & \longrightarrow & \cdots & \longrightarrow & F^{n-1} & \longrightarrow & F^0 \otimes O(p)
\end{array}$$

where the commutativity of the final square follows from the commutativity of:

$$\begin{array}{ccc} E_0 & \xrightarrow{n_E} & E_0 \otimes O(p) \\ \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f_0 \otimes id_{O(p)}} \\ F^0 & \xrightarrow{n_F} & F^0 \otimes O(p) \end{array}$$

where n_E and n_F are natural morphisms. Denote by $F^\bullet[1]$ the shifting of $F^\bullet$ by 1:

$$F^1 \longrightarrow F^2 \longrightarrow \cdots \longrightarrow F^0 \otimes O(p) \longrightarrow F^1 \otimes O(p)$$

By the same process we get a morphism $f^\bullet : E_1^\bullet \to F^\bullet[1]$, shifting down to $f^\bullet : E_1^\bullet[-1] \to F^\bullet$. This gives $\tilde{f}_1^\bullet : \tilde{E}_1^\bullet \to F^\bullet$ which is surjective at index 1 of point p.
Applying the same process, we set:

$$(\tilde{f}_i^\bullet : \tilde{E}_i^\bullet \to F^\bullet)^{0 \leq i \leq w_p - 1}$$

with $\tilde{f}_i$ surjective onto F^i
Therefore, taking the direct sum we get:

$$d : \bigoplus_{i=0}^{w_p-1} \tilde{E}_i^\bullet \longrightarrow\!\!\!\!\rightarrow F^\bullet$$

giving the required epimorphism. This generalises trivially when we have parabolic structure at more than one point. ☐

Theorem 5.4. *The category* $\mathrm{PVec_w}(X)$ *is quasi-abelian and* $\mathcal{LH}(\mathrm{PVec_w}(X))$ *is equivalent to the category* $\mathrm{PCoh_w}(X)$.

Proof. By Theorem 5.2, $(\mathrm{PTCoh}_w(X), \mathrm{PVec}_w(X))$ form a torsion pair in the abelian category $\mathrm{PCoh}_w(X)$. This implies, by Theorem 3.9, that $\mathrm{PVec}_w(X)$ is quasi-abelian.
By Theorem 5.3, $(\mathrm{PTCoh}_w(X), \mathrm{PVec}_w(X))$ is cotilting. Thus, by Theorem 4.2, $\mathcal{LH}(\mathrm{PVec}_w(X)) \approx \mathrm{PCoh}_w(X)$ ☐

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References

[1] Jyun-Ao Lin, *Spherical Hall algebras of a weighted projective curve*, (2014), arXiv:1410.0896.

[2] Jean-Pierre Schneiders, *Quasi-abelian categories and sheaves*, Mm. Soc. Math. Fr. (N.S.) **1999**, no. 76, vi+134 pp.

[3] A. Bondal and M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

[4] Jochen Heinloth, *Coherent sheaves with parabolic structure and construction of Hecke eigensheaves for some ramified local systems*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 7, 2235–2325 (2005).

[5] Saunders MacLane. *Categories for the working mathematician.* Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, (1971)