The homotopy theory of the little n-discs operad

Master Thesis

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Abstract

In this thesis, we describe the simplicial analogue of the coendomorphism operad of Moreno-Fernández and Wierstra, and define coalgebras in the category of simplicial sets. We show that simplicial n-fold suspensions are coalgebras up to homotopy over the Barratt-Eccles E_n–operad. We also compute an explicit model for the A_∞–operad in simplicial sets, and describe the Boardman-Vogt resolution of the Barratt–Eccles E_n-operad.

Contents

Chapter 1

Introduction

1.1 Préambule

L'opérade $\mathbb{D}_n$ de petits n-disques a été introduit pour la première fois par J. P. May dans son livre de 1972 *The Geometry of Iterated Loop Spaces* [22], bien que cela ait été préfiguré dans les travaux de Stasheff et Boardman-Vogt. Il avait remarqué que les espaces des lacets itérés n-fois portent une structure naturelle monoïdale (jusqu'à l'homotopie) induite par la concaténation de lacets. Il a inventé des opérades afin de capturer cette structure sous-jacente, sans référence à l'espace lui-même. Cette approche a prouvé son utilité immédiatement, quand il a pu montrer que toute algèbre sur $\mathbb{D}_n$ est faiblement homotope à un espace de lacets itérés n-fois, un résultat très célèbre connu sous le nom *May's recognition principle*. Depuis lors, cet opérade a informé beaucoup de progrès en topologie algébrique. Par exemple, on peut montrer que l'homologie de l'opérade de petits n-disques est l'opérade de Poisson paramétrée $Pois_n$ dans les complexes de chaînes [6]. Cela implique immédiatement que l'homologie des espaces de lacets itérés n-fois possède non seulement le produit Pontryagin, mais aussi un produit binaire de degré $1-n$ appelé le crochet de Browder. Les opérations Dyer-Lashof et Kudo-Araki sur la cohomologie avec coefficients dans Z_p des espaces de lacets itérés peuvent être construites par des considérations plus complexes. [9].

Le principe de la dualité de Eckmann-Hilton suggère que les suspensions itérées devraient posséder également une théorie très sympa. Un nouveau article de Moreno-Fernández et Wierstra [25] a étudié cela. Leur approche est, pour chaque espace topologique X, de définir *l'opérade de coendomorphisme* $\mathrm{CoEnd}(X)$ de X (voir Definition 2.3.10). Une coalgèbre de l'opérade $\mathscr{P}$ est défini comme une paire (X, φ) où X est un espace et φ est un morphisme opéradique $\mathscr{P} \to \mathrm{CoEnd}(X)$. Ils montrent que les suspensions itérés n-fois sont des coalgèbres de $\mathbb{D}_n$ et décrivent le double Eckmann-Hilton du crochet de Browder, qui se révèle être une coopération sur le rationnel groupes d'homotopie.

Dans cette thèse, nous étendons une partie de cette théorie au domaine des ensembles simpliciaux. En particulier, nous utilisons le foncteur Ex^∞ de Kan pour définir un (petit) opérade de coendomorphisme (voir Definition 5.1.9) pour tout ensemble simplicial X avec seulement un nombre fini de simplices non-dégénérées, et nous l'utilisons pour définir les coalgebras comme dans le dernier paragraphe. Nous établissons que les suspensions simpliciales itérés n-fois sont des coalgèbres (jusqu'à l'homotopie) de l'opérade E_n de Barratt-Eccles (voir Théorème 5.2.1).

Un objectif secondaire de cette thèse est de comprendre la résolution de Boardman-Vogt. Pour résumer, donné une catégorie de modèle fermée, les opérades réduites sur elle possèdent souvent une structure de modèle induite (voir Théorème 4.1.12). La résolution Boardman-Vogt (voir Definition 4.3.10) est un foncteur de remplacement cofibrant au sein de cette structure. Ceci est important parce que en général, nous définissons les algèbres homotopiques sur une opérade $\mathscr{P}$ comme des algèbres ordinaires sur un remplacement cofibrant de $\mathscr{P}$. La résolution Boardman-Vogt est la manière la plus générale de le faire, mais elle est normalement très grande et souvent pas le moyen le plus efficace de le faire. Dans les complexes de chaînes, par exemple, on préfère normalement travailler avec la résolution Koszul qui est beaucoup plus petite [12].

Dans la dernière partie du chapitre 4 de cette thèse, nous étudions quelques exemples de la résolution Boardman-Vogt dans la catégorie des ensembles simpliciaux. Nous montrons que la composante dans arity n de la résolution Boardman-Vogt de l'opérade associatif simplicial est constituée de $n!$ copies disjointes d'un ensemble simplicial de lié à un associahedron. Nous donnons également une description explicite de la résolution de Boardman-Vogt de l'opérade E_n de Barratt-Eccles.

Il y a deux sujets non standard traités en détail dans ce note. Le premier est le foncteur Ex^∞, étudié pour la première fois par Dan Kan [16] en 1957. Il nous fournit un foncteur de remplacement de fibrant complètement combinatoire dans la structure du modèle Kan-Quillen. Elle permet, entre autres choses, de définir la structure du modèle de Kan-Quillen la structure du modèle de Kan-Quillen sans passer aux espaces topologiques ou invoquant les structures du modèle de Cisinski [26].

Le deuxième sujet spécial est l'opérade E_n de Barratt-Eccles. C'est un modèle particulièrement sympa pour la petite opérade de n-disques dans la catégorie des ensembles simpliciaux. Pour $n = \infty$, cela a été défini pour la première fois par Barratt et Eccles [1] en 1974, le cas oú n est fini ayant été étudié pour la première fois par Smith dans sa thèse de 1981 [29]. Plus célèbre encore, la version dg-algèbre de cet opérade a été utilisée pour donner une preuve de la *Deligne conjecture* ([23], [3]), qui déclare que la complexe Hochschild d'une algèbre associative a naturellement la structure d'une algèbre de l'opérade $\mathbb{D}_2$.

La structure de cette thèse

Trois principaux outils techniques sont utilisés dans ce rapport; catégories de modèles, opérades et ensembles simplicial. Celles-ci sont examinées brièvement dans les sections 2.1, 2.3 et 3.1 respectivement. Nous supposons également une certaine familiarité avec les notions élémentaires de topologie algébrique telles que les complexes CW et les groupes d'homotopie.

Le chapitre 2 de ce note est largement à propos les espaces topologiques et sert principalement de motivation pour le reste de la thèse. La section 2.1 rappelle la définition et les propriétés de base des catégories de modèles et définit la structure du modèle sur la catégorie des espaces topologiques. La section 2.2 rappelle la définition de l'espace et de la suspension de la lacet topologique et explique pourquoi il s'agit de doubles Eckmann-Hilton. Il contient également un bref traitement des produits smash et wedge. La section 2.3 contient un bref aperçu de la théorie de l'opérade, ainsi qu'un examen de la notion de Moreno-Fernández - Wierstra d'une coalgèbre topologique. La section 2.4 donne un traitement de l'opérade $\mathbb{D}_n$ de petits n -disques $\mathbb{D}_n$ et May's recognition principle, tout en incluant la preuve de Moreno-Fernández et Wierstra que les suspensions itérés n-fois sont des coalgèbres sur $\mathbb{D}_n$.

Le chapitre 3 se concentre sur la catégorie des ensembles simpliciaux (et plus tard sur les opérades). La section 3.1 rassemble les définitions de base. La section 3.2 définit la structure du modèle sur les ensembles simpliciaux et étudie l'équivalence Quillen entre $\mathrm{Set}_\triangle$ et les espaces topologiques. Nous fournissons un compte rendu complet du foncteur Ex^∞ de Kan dans la section 3.4. Enfin, dans la section 3.5, nous traitons l'opérade E_n de Barratt-Eccles et montrons que sa réalisation géométrique est faiblement homotopique à l'opérade de petits n-disques.

Le chapitre 4 traite de la catégorie modèle d'opérades. La section 4.1 utilise le principe de transfert pour munir la catégorie d'opérades réduites d'une catégorie de modèle (suffisamment agréable) avec une structure de modèle. La section 4.2 rappelle la construction de l'opérade libre. Cela utilise beaucoup des mêmes idées qui seront utilisées dans la section 4.3, où nous définissons un foncteur de remplacement des cofibrants, appelé la résolution Boardman-Vogt, dans la catégorie modèle d'opérades que nous avons construites dans la section 4.1. Enfin, nous concluons le chapitre avec un petit nombre d'exemples de la résolution Boardman-Vogt. Dans la sous-section 4.1.1, nous voyons comment la résolution Boardman-Vogt de l'opérade associatif dans la catégorie des espaces topologiques donne naissance aux polytopes de Stasheff. Les sous-sections 4.4.2 et 4.4.3 contiennent nos premiers résultats originaux, quoique faciles, de cette thèse - des descriptions concrètes des résolutions Boardman-Vogt de l'opérade associatif et de l'opérade E_n de Barratt-Eccles dans des ensembles simpliciaux.

Le chapitre 5 contient nos principaux résultats. Dans la section 5.1, nous trouvons un petit modèle simplicial pour l'opérade de coendomorphisme de Moreno-Fernández et Wierstra. Dans la section 5.2, nous montrons que les suspensions simpliciales itérés n-fois sont des coalgèbres sur l'homotopie opérade E_n de Barratt-Eccles.

1.2 Préambule (version anglaise)

The little n-discs operad $\mathbb{D}_n$ was first introduced by J. P. May in his 1972 book *The Geometry of Iterated Loop Spaces* [22], although it was foreshadowed in the work of Stasheff and Boardman-Vogt. He had noticed that n-fold loop spaces carry a natural monoidal (up to homotopy) structure induced by concatenation of loops. He invented operads in order to capture this underlying structure without reference to the space itself. This approach proved its utility immediately, when he was able to show that any algebra over $\mathbb{D}_n$ is weakly homotopic to an n-fold loop space, a famous result known as May's recognition principle. Since then, this operad has informed much progress in algebraic topology. For example, it can be shown that the homology of the little n-discs operad is the parameterized Poisson operad $Pois_n$ in chain complexes [6]. This immediately implies that the homology of n-fold loop spaces possesses not just the Pontryagin product induced by the concatenation of loops, but also a binary product of degree $1 - n$ called the Browder bracket. The Dyer-Lashof and Kudo-Araki operations on the mod p cohomology of iterated loop spaces may be constructed by more complex considerations [9].

The principle of Eckmann-Hilton duality suggests that that iterated suspensions should possess a similarly rich theory. A recent preprint of Moreno-Fernández and Wierstra [25] has studied this. Their approach is, for each topological space X, to define the *coendomorphism operad* $\mathrm{CoEnd}(X)$ (Definition 2.3.10). A $\mathscr{P}$-coalgebra is defined to be a pair (X, φ) where X is a space and φ is an operadic morphism $\mathscr{P} \to \mathrm{CoEnd}(X)$. They show that n-fold suspensions are $\mathbb{D}_n$-coalgebras and describe the Eckmann-Hilton dual of the Browder bracket, which turns out to be a cooperation on the rational homotopy groups.

In this thesis, we extend some of this theory to the realm of simplicial sets. In particular, we use Kan's Ex^∞ functor to define a small coendomorphism operad (Definition 5.1.9) for any simplicial set X with only finitely many non-degenerate simplices, and use this to define coalgebras in the same way as in the last paragraph. We establish, via model theoretic arguments, that n-fold simplicial suspensions are coalgebras (up to homotopy) of the Barratt-Eccles E_n operad (Theorem 5.2.1).

A secondary goal of this thesis is to understand the Boardman-Vogt resolution. To summarise, given a closed model category, the reduced operads over it often possess an induced model structure (see Theorem 4.1.12). The Boardman-Vogt resolution (Definition 4.3.10) is a cofibrant replacement functor within this structure. This is important because in general we define homotopy algebras over an operad $\mathscr{P}$ as ordinary algebras over a cofibrant replacement of $\mathscr{P}$. The Boardman-Vogt resolution is the most general way of doing this, but it is normally very large and often not the most efficient way to do things. In chain complexes, for example, one normally prefers to work with the far smaller Koszul resolution [12].

In the closing section of Chapter 4 of this thesis, we study some examples of the Boardman-Vogt resolution in the category of simplicial sets. We show that the arity n component of the Boardman-Vogt resolution of the simplicial associative operad consists of $n!$ disjoint copies of a simplicial set analogous to an associahedron. We also give a similarly explicit description of the Boardman-Vogt resolution of the Barratt-Eccles E_n-operad.

There are two nonstandard topics given complete expository treatment in this report. The first of these is the Ex^∞ functor, originally studied by Dan Kan [16] in 1957. It provides us with a completely combinatorial fibrant replacement functor in the Kan-Quillen model structure. Among other things, it allows one to define the Kan-Quillen model structure passing to topological spaces or invoking Cisinski model structures [26].

The second special topic is the Barratt-Eccles E_n operad. This is a particularly nice model for the little n-discs operad in the category of simplicial sets. For $n = \infty$, this was first defined by Barratt and Eccles [1] in 1974, with the finite n case first studied by Smith in his 1981 PhD thesis [29]. Most famously, the dg-algebra version of this operad has been used to give a proof of the *Deligne conjecture* ([23], [3]), which states that the Hochschild complex of an associative algebra naturally has the structure of $\mathbb{D}_2$-algebra.

The structure of this thesis

There are three major technical tools used in this report; model categories, operads and simplicial sets. These are briefly reviewed in Sections 2.1, 2.3 and 3.1 respectively. We also assume some familiarity with elementary notions of algebraic topology such as CW-complexes and homotopy groups.

Chapter 2 of this report is broadly themed around topological spaces, and mainly serves as motivation for the remainder of the thesis. Section 2.1 recalls the definition and basic properties of model categories, and defines the model structure on the category of topological spaces. Section 2.2 recalls the definition of topological loop space and suspension, and explains why they are Eckmann-Hilton duals. It also contains a brief treatment of smash and wedge products. Section 2.3 contains a brief overview of operad theory, as well as reviewing the Moreno-Fernández–Wierstra notion of a topological coalgebra. Section 2.4 gives a treatment of the little n-discs operad $\mathbb{D}_n$ and May's recognition principle, while also including Moreno-Fernández–Wierstra's proof that n-fold suspensions are coalgebras (up to homotopy) over $\mathbb{D}_n$.

Chapter 3 focuses on the category of simplicial sets (and later the operads over it). Section 3.1 collates the basic definitions. Section 3.2 defines the model structure on simplicial sets and studies the Quillen equivalence between $\mathrm{Set}_{\triangle}$ and topological spaces. We provide a comprehensive account of Kan's Ex^{∞} functor in Section 3.4. Finally, in Section 3.5, we treat the Barratt-Eccles E_n-operad and show that its geometric realization is weakly homotopic to the little n-discs operad.

Chapter 4 is about the model category of operads. Section 4.1 uses the Transfer Principle to endow the category of reduced operads over a suitably nice model category with a model structure. Section 4.2 recalls the construction of the free operad. This uses a lot of the same ideas that shall be used in Section 4.3, where we define a cofibrant replacement functor, called the Boardman-Vogt resolution, in the model category of operads we constructed in Section 4.1. Finally, we conclude the chapter with a small number of examples of the Boardman-Vogt resolution. In Subsection 4.1.1 we see how the Boardman-Vogt resolution of the the associative operad in the category of topological spaces gives rise to Stasheff polytopes. Subsection 4.4.2 and 4.4.3 contain the first original, albeit easy, results of this thesis - concrete descriptions of the Boardman-Vogt resolutions of the associative and Barratt-Eccles operads in simplicial sets.

Chapter 5 contains our main results. In Section 5.1, we find a small simplicial model for the coendomorphism operad of Moreno-Fernández and Wierstra. In Section 5.2, we show that n-fold simplicial suspensions are coalgebras over the homotopy Barratt-Eccles E_n-operad.

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Chapter 2

Topological Spaces

This chapter is a hodgepodge of preliminaries for the rest of this report. A major goal of this thesis is to study simplicial (a word that can be read by the uninitiated as *combinatorial*) models for operads in topological space. This chapter is largely about introducing these constructions and methods in their native habitat and shedding a little light on how they behave there.

We shall begin with a barebones study of model categories. This is the natural language of homotopy theory and we shall make heavy use of it throughout this report. The main reason why simplicial sets are useful is that the category they form in a model theoretic sense is equivalent to the category of topological spaces up to homotopy; we say it *models* it.

We shall follow this up with a section defining a few topological constructions. These will include the wedge and smash products, and the suspension and loop space constructions. In the spirit of brevity, we do not fully explain the motivation for these constructions, which lie in the theory of spectra. The reader may however choose to view them as elementary enough to be of interest in their own right.

Finally, following a review of the definition of an operad, we shall study the little n-discs operad $\mathbb{D}_n$, the principal protagonist of this report. We shall see how loop spaces possess a natural $\mathbb{D}_n$-algebra structure. The converse to this statement is the celebrated recognition principle of J.P. May, which we shall formally state.

There is a philosophy in modern mathematics that has motivated a lot of research called Eckmann-Hilton duality. In brief, in homotopy theory we think of certain ideas as having a natural partner. For example, we have the following natural pairings: homotopy groups $\sim$ cohomology groups; wedge products $\sim$ direct products; suspensions $\sim$ loop spaces. Eckmann-Hilton duality asserts that if all concepts in a given theorem are replaced with their partner then that theorem should remain true. Dual theorems do not necessarily admit dual proofs, and that is why Eckmann-Hilton duality remains a philosophy and not a result.

In the closing section, we shall study coalgebras over the little n-discs operad. Moreno–Fernández and Wierstra recently showed suspensions are examples of such coalgebras. Eckmann-Hilton duality suggests that this should give rise to a recognition principle for suspensions, although this remains unproven at the current time.

2.1 Model categories

Model categories were originally defined by Dan Quillen [27] to avoid the set-theoretical issues that arise when passing to the localization of a category with respect to some set of its morphisms. As of such, they provide the natural environment in which to do homotopy theory, where we want to invert the *weak homotopy equivalences* (defined below in Example 2.1.4). Classically, doing this allows us to pass from the space of all topological spaces, some of which are horrendous to work with, to the well-behaved full subcategory of CW-complexes. The first goal of this section is to paint a picture of how this works, by providing the relevant definitions and stating the relevant theorems. Following that, we shall move on to the second goal of this section, which is to define the correct notion of equivalence for model categories. This will be important for us in Section 2.2 where we shall see that topological spaces and simplicial sets both possess model structures that are equivalent in this sense.

A more comprehensive treatment of the following material may be found in [8].

Definition 2.1.1 A *model category* is a category $\mathbf{C}$ equipped with the following three classes of morphisms.

1. *Weak equivalences* $\mathcal{W}$ which we will denote by $\xrightarrow{\sim}$.
2. *Cofibrations* $\mathcal{C}$ which we will denote by $\hookrightarrow$. A cofibration that is also a weak equivalence is called an *acyclic cofibration.*
3. *Fibrations* $\mathcal{F}$ which we will denote by $\twoheadrightarrow$. A fibration that is also a weak equivalence is called an *acyclic fibration.*

These morphisms satisfy the following five axioms.

1. $\mathbf{C}$ contains all small limits and colimits.
2. If it is possible to compose the morphisms f and g, then if any two of $f, g, f \circ g$ is in $\mathcal{W}$ the third is. This is sometimes called the '2-out-of-3' rule.

3. If f is a retract of g and $g \in \mathcal{W}$ (or $\mathcal{C}$ or $\mathcal{F}$) then $f \in \mathcal{W}$ (or $\mathcal{C}$ or $\mathcal{F}$). In other words, if we have a commutative diagram

$$\begin{array}{ccccc} A & \stackrel{i}{\to} & B & \stackrel{r}{\to} & A \\ f\downarrow & & g\downarrow & & f\downarrow \\ A' & \stackrel{j}{\longrightarrow} & B' & \stackrel{t}{\longrightarrow} & A' \end{array}$$

with $ri = \mathrm{id}_A$ and $tj = \mathrm{id}_{A'}$ such that g is any one of the three classes of morphism, then f is a member of the same class.

4. Consider the following commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

- If $i \in \mathcal{C}$ and $p \in \mathcal{W} \cap \mathcal{F}$ (ie. p is an acyclic fibration) then the dotted lift exists.
- If $i \in \mathcal{C} \cap \mathcal{W}$ (ie. i is an acyclic cofibration) and $p \in \mathcal{F}$ then the dotted lift exists.

5. Every morphism $f : X \to Y$ admits two factorisations both functorial in f

$$X \overset{\sim}{\hookrightarrow} P_f \twoheadrightarrow Y \qquad X \hookrightarrow C_f \overset{\sim}{\twoheadrightarrow} Y$$

In other words, every morphism in $\mathbf{C}$ has admits two decompositions. The first is into a fibration followed by an acyclic cofibration, the second is into a acyclic fibration followed by a cofibration. Moreover these decompositions are functorial in the arrow category of $\mathbf{C}$.

Definition 2.1.2 We say that an object $X \in \mathbf{C}$ is *fibrant* if its terminal morphism is a fibration and *cofibrant* if its initial morphism is a cofibration.

Remark 2.1.3 By the fifth model category axiom, given any object X in a model category it is possible to find both *cofibrant and fibrant replacements* for it. A fibrant replacement for X is an object $R(X)$ such that

$$X \overset{\sim}{\hookrightarrow} R(X) \twoheadrightarrow *$$

where $*$ is the terminal object in $\mathbf{C}$. Note in particular that $R(X)$ is a fibrant object that is weakly equivalent to X. A cofibrant replacement is the dual notion, where we factorise the morphism from the initial object through a cofibrant object weakly equivalent to X. By carrying out these two procedures consecutively it is possible to find an object that is simultaneously fibrant, cofibrant and weakly equivalent to X.

The first example that we are about to give is very important. It, together with the Kan-Quillen structure on $\mathrm{Set}_{\triangle}$, will form one of the two major examples of model categories that we shall see in this report.

Example 2.1.4 [8, Proposition 8.3] In the category Top, a *Serre fibration* is a continuous function $p : X \to Y$ such that, for all commutative diagrams of the form

$$\begin{array}{ccc} [0,1]^n \times \{0\} & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ [0,1]^n \times [0,1] & \longrightarrow & Y. \end{array}$$

the dotted lift exists. A *weak homotopy equivalence* is a continuous function $f : X \to Y$ such that, for each $n \in \mathbb{N}$, the induced morphism of homotopy groups $f_n^* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism. We define the *Quillen model structure on* Top to have weak homotopy equivalences for weak equivalences, Serre fibrations for fibrations and inclusions of generalised CW–complexes for cofibrations. Two facts about this model structure are worth noting. Firstly, all objects are fibrant. Secondly, every topological space is weakly (but **not** necessarily strongly) equivalent to a CW–complex.

Next, we recall what it means to localize a category with respect to some class of morphisms.

Definition 2.1.5 Let C be a category, and $W \subseteq C$ be a class of morphisms. A functor $F : C \to D$ is said to be the *localization functor* of C with respect to W if

- $F(f)$ is an isomorphism for each $f \in W$.
- whenever $G : C \to D'$ is a functor carrying elements of W to isomorphisms, there exists a unique functor $G' : D \to D'$ such that $G'F = G$.

We normally use the notation $C[W^{-1}]$ to denote D, and call it the *localization of D with respect to W*.

The homotopy category of a model category is just its localization with respect to the class of weak equivalences.

Definition 2.1.6 Let $(\mathbf{C}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a model category. The *homotopy category* of $\mathbf{C}$ is defined as

$$\mathrm{Ho}(\mathbf{C}) := \mathbf{C}[\mathcal{W}^{-1}].$$

Example 2.1.7 [8, Proposition 8.4] If Top is equipped with the Quillen model structure, the homotopy category is equivalent to the full subcategory consisting of CW–complexes.

By itself Definition 2.1.6 is not very useful, and the astute reader has probably noticed that there is no meaningful reference made to the morphism classes $\mathcal{C}$ and $\mathcal{F}$. We turn now to the more interesting problem of how to compute such a thing; this requires introducing the notion of a homotopy. For the rest of this section we will assume that **C** is a model category.

Definition 2.1.8 A *cylinder* of $A \in \mathbf{C}$ is a factorisation

$$A \sqcup A \xrightarrow{(i_0,i_1)} P \xrightarrow{\sim} A$$

of the canonical function $A \sqcup A \to A$ into a weak equivalence followed by a cofibration.

Definition 2.1.9 Let $f, g : A \to X$ be two morphisms. A (left) homotopy between f and g, written $f \sim g$, is the data of a cylinder $A \sqcup A \xrightarrow{(i_0,i_1)} P \xrightarrow{\sim} A$ and a function $H : P \to X$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

Example 2.1.10 In the category of topological spaces equipped with the Quillen model structure, $A \times I$ is a natural choice of cylinder. The map i_0 is given by $a \mapsto (a, 0)$ for all $a \in A$ and similarly i_1 is $a \mapsto (a, 1)$. In this way, a left homotopy coincides with the usual idea of a homotopy in Top .

Remark 2.1.11 The notion of a right homotopy between two morphisms also exists and, in general, is not equivalent to the notion of a left homotopy. However the only situation that we shall consider in this note is that of the next lemma. In this very special case, left and right homotopies are the same.

Lemma 2.1.12 *[8, Lemma 4.7] Let A be a cofibrant object and X be a fibrant object. Then $\sim$ is an equivalence relation on* $\mathrm{Hom}_{\mathbf{C}}(A, X)$.

Theorem 2.1.13 *[8, Proposition 5.11] Let $\pi\mathbf{C}_{cf}$ be the category consisting of objects of* **C** *that are both fibrant and cofibrant and which has for morphisms*

$$\mathrm{Hom}_{\pi\mathbf{C}_{cf}}(A, X) := \mathrm{Hom}_{\mathbf{C}}(A, X)/\sim .$$

Then $\mathrm{Ho}(\mathbf{C})$ *is equivalent to* $\pi\mathbf{C}_{cf}$.

When we are working with model categories usually the only information of intrinsic interest is that that exists on the level of homotopy categories. The correct notion of equivalence therefore is not therefore an equivalence of categories, but an equivalence of **homotopy categories.** We are going to formalise this idea. The first thing to do is to show how some functors on model categories induce functors on underlying homotopy categories.

Definition 2.1.14 Let **C** and **D** be a model categories. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and let $\lambda_{\mathbf{C}} : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ be the localization functor.

- A *left derived functor* of F is a functor $\mathbb{L}F : \text{Ho}(\mathbf{C}) \to \mathbf{D}$ and a natural transformation $\alpha : \mathbb{L}F \circ \lambda_{\mathbf{C}} \Rightarrow F$ satisfying the following universal property: for every pair $(G : \text{Ho}(\mathbf{C}) \to \mathbf{D}, \beta : G \circ \lambda_{\mathbf{C}} \Rightarrow F)$ there exists a unique natural transformation $\theta : G \to \mathbb{L}F$ such that β is the composite

$$G \circ \lambda \xRightarrow{\theta \circ \lambda} \mathbb{L}F \circ \lambda \xRightarrow{\alpha} F$$

  A *total left derived functor* is a left derived functor of $\lambda_{\mathbf{D}} \circ F : \mathbf{C} \to \mathbf{D} \to \text{Ho}(\mathbf{D})$.

- A *right derived functor* of F is a functor $\mathbb{R}F : \text{Ho}(\mathbf{C}) \to \mathbf{D}$ and a natural transformation $\varepsilon : \mathbb{R}F \circ \lambda_{\mathbf{C}} \Rightarrow F$ satisfying the following universal property: for every pair $(G : \text{Ho}(\mathbf{C}) \to \mathbf{D}, \sigma : G \circ \lambda_{\mathbf{C}} \Rightarrow F)$ there exists a unique natural transformation $\theta : G \to \mathbb{R}F$ such that σ is the composite

$$F \xRightarrow{\varepsilon} \mathbb{R}F \circ \lambda_{\mathbf{C}} \xRightarrow{\theta \circ \lambda_{\mathbf{C}}} G \circ \lambda$$

  A *total right derived functor* is a right derived functor of $\lambda_{\mathbf{D}} \circ F : \mathbf{C} \to \mathbf{D} \to \text{Ho}(\mathbf{D})$.

Remark 2.1.15 If a (total) left or right derived functor exists (which it need not) it is unique up to isomorphism. It is customary to abuse notation by denoting total derived functors by $\mathbb{L}F$ and $\mathbb{R}F$, a convention we will adopt henceforth.

Since total derived functors of a general functor $F : \mathbf{C} \to \mathbf{D}$ need not exist we must find a class of functors for which they certainly do. The following definition and proposition give us exactly what we need.

Definition 2.1.16 Let $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$ be an adjunction between two model categories. If any of the four following equivalent notions are satisfied we say that the adjunction is *Quillen*:

- F preserves both cofibrations and acyclic cofibrations.
- G preserves fibrations and acyclic fibrations.
- F preserves cofibrations and G preserves fibrations.
- F preserves acyclic cofibrations and G preserves acyclic fibrations.

Remark 2.1.17 Of course that all four definitions are equivalent should be proven but since this section is an overview we shall omit this. In any case the relevant implications all have similar proofs and follow straightforwardly from the fourth model category theory axiom. The interested reader can find the full details in [14].

Proposition 2.1.18 *[8, Theorem 9.7 (i)] If $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$ is a Quillen adjunction then $\mathbb{L}F$ and $\mathbb{R}G$ exist and form a adjunction*

$$\mathbb{L}F : \mathrm{Ho}(\mathbf{C}) \leftrightarrows \mathrm{Ho}(\mathbf{D}) : \mathbb{R}G.$$

We are finally in a position to define the desired notion of equivalence.

Definition 2.1.19 A Quillen adjunction $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$ is called a *Quillen equivalence* if the induced adjunction of homotopy categories is an equivalence of categories.

Remark 2.1.20 [8, Theorem 9.7 (ii)] In practice when we want to show that a Quillen adjunction $F : \mathbf{C} \leftrightarrows \mathbf{D} : G$ is a Quillen equivalence it suffices to show that any one of the following conditions holds;

- If $A \in \mathbf{C}$ is a cofibrant object and $X \in \mathbf{D}$ is fibrant, a morphism $F(A) \to X$ is a weak equivalence if and only if its adjoint morphism $A \to G(X)$ is.
- For every cofibrant object $c \in \mathbf{C}$, the composite $c \xrightarrow{\mu_c} GF(c) \to G(F(c)^{fib})$, of the adjunction unit with a fibrant replacement $F(c) \to F(c)^{fib}$, is a weak equivalence in $\mathbf{C}$. We also require that for every fibrant object $d \in \mathbf{D}$, the composite $F(G(d)^{cofib}) \to F(G(d)^{cofib}) \xrightarrow{\varepsilon_d} d$, of the cofibrant replacement functor $G(d)^{cofib} \to G(d)$ with the counit of the adjunction, is a weak equivalence in $\mathbf{D}$.

2.2 Classical constructions

In this section we collect the classical topology necessary to read this report. First we introduce two classical topological constructions; loop spaces and suspensions. These are both extremely important in stable homotopy theory, and hence central to the story of algebraic topology itself. We will also study the category of pointed topological spaces, where we shall meet two natural binary operations, the smash and wedge products. The first of these is the category-theoretical product for pointed spaces, while the second is the coproduct, and will be needed to define the topological coendomorphism operad in Section 2.3.

In this thesis, we will use I to denote the unit interval $[0,1]$ and D^n to denote the unit n-disc. We shall also identify the unit n-sphere S^n with $D^n/\partial D^n$. In this way it is equipped with a natural choice of base point. All spaces that we shall work with in this section are assumed to have the homotopy type of a CW-complex.

Definition 2.2.1 Let X and Y be two topological spaces. We equip their hom-set

$$\mathrm{Hom}_{\mathrm{Top}}(X,Y)$$

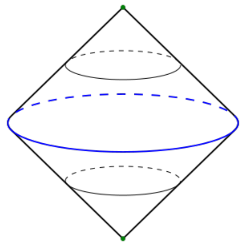


Figure 2.1: The unreduced suspension (image credit to [32])

with the compact open topology so that it forms a topological space. This is called the *mapping space from* X *to* Y and is denoted as either Y^X or $\mathrm{Map}(X, Y)$.

Remark 2.2.2 The mapping space is an example of the more general phenomenon of an *inner hom*, which we shall see more in Chapter 3 (see Definition 4.1.1). The main property of mapping spaces is that for all spaces Y, the functor $\mathrm{Map}_{\mathrm{Top}}(Y, -)$ is right adjoint to the functor $- \times Y$. In other words, for all spaces X and Z, there is a natural bijection of sets

$$\mathrm{Hom}_{\mathrm{Top}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathrm{Top}}(X, \mathrm{Map}_{\mathrm{Top}}(Y, Z)).$$

Definition 2.2.3 Let $(X, *)$ be a pointed topological space. Then the *loop space* of X is the set of pointed continuous maps from the pointed circle to X

$$\Omega X := \mathrm{Hom}_{\mathrm{Top}_*}((S^1, *), (X, *))$$

equipped with the compact-open topology. We view this as a pointed space with the distinguished point being the constant map $S^1 \to *$. The *n-fold loop space* of X is simply $\Omega^n(X) = \Omega(\Omega^{n-1}(X))$ where $\Omega^1 X$ is ΩX.

Definition 2.2.4 The *reduced cone* CX of a pointed space X is $(X \times I)/(X \times \{0\} \cup * \times I)$ The *reduced suspension* ΣX of a pointed space X is $(X \times I)/(X \times \{0\} \cup * \times I \cup X \times \{1\})$. If we say that $\Sigma^1 X := \Sigma X$, then the *n-fold reduced suspension* $\Sigma^n(X)$ is $\Sigma(\Sigma^{n-1}(X))$.

Remark 2.2.5 There are more geometrically intuitive notions of an unreduced cones and unreduced suspensions defined respectively as

$$uCX = (X \times I)/(X \times \{0\})$$

$$u\Sigma X := (X \times I)/(X \times \{0\} \cup X \times \{1\}).$$

The unreduced suspension is visualised in Figure 2.1. To obtain the reduced suspension we also contract the line $* \times I$ to a point. In future we will only ever work with reduced suspensions so we will drop the qualifier and refer to them as *suspensions*.

Loop spaces and suspensions are actually adjoint functors in the enriched category. This result is a basic example of what is known in the literature as Eckmann-Hilton duality.

Proposition 2.2.6 *Let X, Y be pointed topological spaces. Then we have*

$$\mathrm{Map}_{Top_*}(\Sigma X, Y) \cong \mathrm{Map}_{\mathrm{Top}_*}(X, \Omega Y).$$

In other words, there is an enriched adjunction with the left adjoint being the suspension functor and the right being the loop space functor.

We are not going to prove this result until we have introduced the smash product, the tensor product in the category of pointed topological spaces. Proposition 2.2.6 will then be an easy application of tensor-hom adjunction.

Definition 2.2.7 Let (A, a_0) and (B, b_0) be two pointed spaces. Their *wedge sum* is $A \vee B := (A \sqcup B)/\sim$ where $\sim$ is the identification $a_0 \sim b_0$ and $\sqcup$ signifies disjoint union.

This corresponds to gluing two pointed topological spaces together at at their distinguished points. For example the wedge sum of two circles is a figure of eight.

Definition 2.2.8 The *smash product* of pointed topological spaces A and B is $A \wedge B := (A \times B)/(A \vee B)$.

Proposition 2.2.9 *The wedge product is distributive over the smash product, ie. if X, Y and Z are pointed topological spaces then $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Y)$.*

Proof Using the definition of the smash product, we have that

$$X \wedge (Y \vee Z) = (X \times (Y \vee Z))/(* \times (Y \vee Z) \cup X \times *).$$

Then, applying the definition of a wedge product to $(Y \vee Z)$ we see that

$$\begin{aligned}&(X \times (Y \vee Z))/(* \times (Y \vee Z) \cup X \times *) = \\ &\qquad (X \times (* \times Z \cup Y \times *))/(* \times (* \times Z \cup Y \times *) \cup X \times *).\end{aligned}$$

Using the distributivity of the Cartesian product over itself, we obtain

$$\begin{aligned}&(X \times (* \times Z \cup Y \times *))/(* \times (* \times Z \cup Y \times *) \cup X \times *) = \\ &\qquad (X \times * \times Z \cup X \times Y \times *)/(* \times * \times Z \cup * \times Y \times * \cup X \times * \times *)\end{aligned}$$

Unions commute with quotients giving

$$\begin{aligned}&(X \times * \times Z \cup X \times Y \times *)/(* \times * \times Z \cup * \times Y \times * \cup X \times * \times *) = \\ &((X \times * \times Z)/(* \times * \times Z \cup X \times * \times *)) \cup ((X \times Y \times *)/(* \times Y \times * \cup X \times * \times *))\end{aligned}$$

Finally, using the definition of the wedge and smash products this contracts to

$$((X \times * \times Z)/(* \times * \times Z \cup X \times * \times *)) \cup \\ ((X \times Y \times *)/(* \times Y \times * \cup X \times * \times *)) = (X \wedge Y) \vee (X \wedge Y).$$

□

Remark 2.2.10 Once can easily show that the smash product is the category-theoretical product in the category of simplicial sets. Moreover, one can easily show that the functor $X \wedge -$ is left adjoint to the functor $\mathrm{Map}_{\mathrm{Top}_*}(X, -)$ for all pointed topological spaces X.

Smash products are also intimately related to suspensions.

Proposition 2.2.11 *Let X be a pointed topological space. Then the suspension ΣX is homeomorphic to $X \wedge S^1$.*

Proof The 1-sphere S^1, which is defined as $D^1/\partial D^1$, is homeomorphic to $I/(\{0\} \cup \{1\})$. Then we have $X \wedge S^1 = (X \times I/(X \times \{0\} \cup X \times \{1\}))/(X \vee S^1) = (X \times I)/(X \times \{0\} \cup * \times I \cup X \times \{1\}) = \Sigma X$. □

With the aid of Proposition 2.2.11 it becomes very easy to prove Proposition 2.2.6.

Proof (Proposition 2.2.6) By Lemma 2.2.11 we have that

$$\mathrm{Map}_{\mathrm{Top}_*}(\Sigma X, Y) \cong \mathrm{Map}_{\mathrm{Top}_*}((X \wedge S^1), Y).$$

By the wedge product–mapping space adjunction, we further have

$$\mathrm{Map}_{\mathrm{Top}_*}((X \wedge S^1), Y) \cong \mathrm{Map}_{\mathrm{Top}_*}(X, \mathrm{Map}_{\mathrm{Top}_*}(S^1, Y)).$$

Finally, once again applying Lemma 2.2.11 we can conclude that

$$\mathrm{Map}_{\mathrm{Top}_*}(X, \mathrm{Map}_{\mathrm{Top}_*}(S^1, Y) \cong \mathrm{Map}_{\mathrm{Top}_*}(X, \Omega Y),$$

thus proving the proposition. □

The last purely topological fact that we will need is the following alternative description of n-fold loop spaces.

Proposition 2.2.12 *There is a homeomorphism between $\Omega^n X$ and $\mathrm{Hom}_{\mathrm{Top}^*}(S^n, X)$.*

Proof We prove this by induction. The base case follows by definition. Next we claim that $\mathrm{Hom}_{\mathrm{Top}^*}(S^{k+1}, X) = \mathrm{Hom}_{\mathrm{Top}^*}(S^k, \mathrm{Hom}_{\mathrm{Top}^*}(S^1, X))$. The left hand side is homeomorphic to $\mathrm{Hom}_{\mathrm{Top}^*}(S^k \wedge S^1, X)$ by tensor-hom adjunction. $S^k \wedge S^1$ is equal to $(D^n \times D^1)/(\partial D^n \times D^1 \cup D^n \times \partial D^1)$. This is homeomorphic to $(I^n \times I)/(\partial I^n \times I \cup I^n \times \partial I)$ This is equal to $I^{n+1}/\partial I^{n+1}$, which is homeomorphic to S^{n+1}, proving the proposition. □

Remark 2.2.13 In the course of proving the last result we showed that S^n is of the same homotopy type as $(S^1)^{\wedge n}$. We could equally apply this to Proposition 2.2.11 to obtain the Hilton-Eckmann dual of the last result. That is, $\Sigma^n X$ is homotopy equivalent to $X \wedge S^n$.

2.3 Operads and algebras

Operads were first introduced in 1971 by J. P. May to aid in the study of iterated loop spaces [22], a goal that motivates us here too. We shall have more to say on iterated loop spaces in Section 2.4, but in this section we will focus on the basic ideas. Intuitively an algebra in a category C is an object of C equipped with some n-ary operations from the object to itself. A well-known example is a vector space equipped with a Lie bracket or an associative operation. An operad is a gadget that captures what we mean by 'Lieness' or 'associativity,' in the latter case in a way that can be made category–independent. More precisely we will define a $\mathscr{P}$-algebra to be a special type of module over an operad $\mathscr{P}$, which we could take as either the Lie or associative operad. Of course we should not lose sight of the fact that we are primarily interested in topological spaces. Our treatment here will be squarely aimed at preparing the ground for the introduction of little n-disk operad, which is the principal protagonist of this report. A more comprehensive treatment of algebraic operads can be found in [19, Chapter 5].

Operads are fairly complicated beasts to describe and so we will break their definition into several steps. We shall start by describing the objects that form the underlying combinatorial data of an operad.

Definition 2.3.1 Let $(\mathbf{C}, \otimes, 1)$ be a symmetric monoidal category. An $\mathbb{S}$-*module* is a family $M = (M(0), M(1), M(2) \dots M(i) \dots)$, where each $M(i)$ is an object of $\mathbf{C}$ equipped with a right $\mathbb{S}_i$-action, where $\mathbb{S}_i$ is the symmetric group. If $\mu \in M(n)$ then μ is said to be of *arity n.*

A *morphism of* $\mathbb{S}$-*modules* $f : M \to N$ is a collection of maps $\{f_n : M(n) \to N(n) : n \in \mathbb{N}\}$. where f_n is $\mathbb{S}_n$-equivariant for all n.

Definition 2.3.2 An *(symmetric) operad* is an $\mathbb{S}$-module $\mathscr{P}$ together with composition maps

$$\gamma : \mathscr{P}(r) \otimes \mathscr{P}(n_1) \otimes \cdots \otimes \mathscr{P}(n_r) \to \mathscr{P}(n_1 + \cdots + n_r)$$

for every $r, n_1, \dots, n_r \geq 0$ and a right action of $\mathbb{S}_r$ on $\mathscr{P}(r)$ which we denote $*$. We further require that these satisfy the following axioms.

1. (Associativity) We have $\gamma(x, \gamma(x_1, x_{1,1}, \dots x_{1,n_1}), \dots, \gamma(x_1, x_{r,1}, \dots x_{r,n_r}))$ is equal to $\gamma(\gamma(x, x_1, \dots, x_r), x_{1,1}, \dots, x_{r,n_r})$ for all $x \in \mathscr{P}(r)$, $x_i \in \mathscr{P}(n_i)$ and $x_{i,j} \in \mathscr{P}(n_{i,j})$.

2. (Unit) We have a distinguished element $1 \in \mathscr{P}(1)$. This satisfies the identity $\gamma(x, 1, \ldots, 1) = \gamma(1, x) = x$.
3. (Equivariance) Let $\sigma \in \mathbb{S}_r$ and $\sigma_i \in \mathbb{S}_{n_i}$. Then we have

$$\gamma(x * \sigma, x_{\sigma(1)}, \ldots, x_{\sigma(r)}) = \gamma(x, x_1, \ldots, x_r) * \sigma_{n_1, \ldots, n_r}$$

$$\gamma(x, x_1 * \sigma_1, \ldots, x_r * \sigma_r) = \gamma(x, x_1, \ldots, x_r) * (\sigma_1 \otimes \cdots \otimes \sigma_r)$$

where $x \in \mathscr{P}(r)$, $x_i \in \mathscr{P}(n_i)$ and $\sigma_{n_1,\ldots,n_r}$ is the permutation that operates on $\{1, 2, \ldots, n_1 + \cdots + n_r\}$ by breaking it into n blocks with the i^{th} of size n_i and permuting these blocks by σ.

An operadic morphism $\Phi : \mathscr{P} \to \mathscr{Q}$ is a morphism of $\mathbb{S}-$modules that is compatible with composition and sends the identity to the identity.

We will briefly explain how one should think about the two previous definitions in the category Vect which will hopefully give the reader a more general intuition. Each $\mathscr{P}(n)$ is a representation of $\mathbb{S}_n$ which we view as the 'space of -arity n formal operations'. We can think of each element of $\mathscr{P}(n)$ as a formal operation that takes n arguments. Evidently given an operation $\alpha(-, -, \cdots, -)$ that takes r arguments and n operations $\beta_i(-, -, \cdots, -)$ each taking n_i arguments we can compose them to produce the operation $\alpha(\beta_1(-, \cdots, -), \ldots, \beta_n(-, \cdots, -))$ which takes $n_1 + \cdots + n_r$ and thus can be regarded as an element of $\mathscr{P}(n_1 + \cdots + n_r)$. This is our function γ. It is clear that this γ should satisfy some notion of associativity. Another nice property one could hope for is the existence of a unary (arity 1) identity operation.

Given an operation one of the most natural things we can do is to permute its inputs. For example, given the arity 3 operation $(x_1 x_2) x_3$ if we apply the permutation (123) to it we get $(x_2 x_3) x_1$. This should be thought of as the action of symmetric group. Equivariance is exactly the property that we need to ensure that composition respects this permutative structure.

Remark 2.3.3 The map γ takes $n + 1$ inputs which can make it quite complex to work with. To simplify formulae we will often replace it with *partial operadic composition*, which takes only 2 inputs. Let $x \in \mathscr{P}(n)$ and $y \in \mathscr{P}(m)$. Then the operation $x \circ_i y \in \mathscr{P}(n + m - 1)$, called the partial composite of x and y at i, is given by

$$x \circ_i y := \gamma(x, 1, \ldots, y, \ldots, 1)$$

where the y occurs at the $i + 1$ place. On the other hand, one can show (see [19, Proposition 5.3.8]) that, for $x \in \mathscr{P}(r)$ and $y_i \in \mathscr{P}(n_i)$ for $r \geq i \geq 1$, that

$$\gamma(x, y_1, \ldots y_r) = (\cdots((x \circ_n y_n) \circ_{n-1} y_{n-1}) \cdots) \circ_1 y_1.$$

To further illustrate the concept of an operad we will give two important examples in the category Vect which we will see analogues of later in Top*.

Example 2.3.4 The *associative operad* Assoc has the underlying $\mathbb{S}$-module given by the regular representation in each arity. This recognizes the fact that we have a different associative operation $x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$ for each $\sigma \in \mathbb{S}_n$. The composition map γ is defined by $\gamma(\sigma, \zeta_1, \dots, \zeta_r) := \pi(\zeta_{\sigma(1)} \times \zeta_{\sigma(2)} \times \cdots \times \zeta_{\sigma(n)})$ where π is the natural embedding $\mathbb{S}_{n_1}\mathbb{K} \times \cdots \times \mathbb{S}_{n_r}\mathbb{K} \hookrightarrow \mathbb{S}_{n_1+\cdots+n_r}\mathbb{K}$

Example 2.3.5 The *commutative operad* Com has the trivial one-dimensional representation in each arity and composition is also trivial. This is due to the fact that there is only one commutative operation of arity n. Because the trivial representation is the terminal object in the category of representations, Com is the terminal object in the category of symmetric operads over sets, spaces and simplicial sets.

Remark 2.3.6 There is a somewhat simpler notion of a *nonsymmetric operad* which is a symmetric operad but without the symmetric structure ie. we do not require that $\mathscr{P}(n)$ be equipped with a right action of $\mathbb{S}_n$ and we do not require equivariance. We will make occasional mention of these. Be warned though that some of the standard notation may appear inconsistent at first. For example, the nonsymmetric associative operad is defined by $\text{Assoc}(n) = *$ for all n. The reason for this effectively boils down to the fact that symmetric operad operations are not independent. In fact they are independent only up to free action of the symmetric group.

The next topic we are going to discuss is the notion of an algebra over an operad. We previously viewed an operad as a structure consisting of formal operations. An algebra over that operad is a structure equipped with concrete realizations of those formal operations via an action.

Definition 2.3.7 Given an object X of a closed symmetric monoidal category M, the *endomorphism operad* $\text{End}(X)$ has underlying $\mathbb{S}-$module structure given by $\mathbf{Hom}(X^{\otimes n}, X)$ in arity n (where **Hom** is the internal hom) equipped with a right $\mathbb{S}_n$-action via permutation of the factors in the tensor product. Operadic composition is given by

$$\gamma : \text{End}(X)(r) \otimes \text{End}(X)(n_1) \otimes \cdots \otimes \text{End}(X)(n_r) \to \text{End}(X)(n_1 + \cdots + n_r)$$
$$(f, f_1, \dots f_n) \mapsto f \circ (f_1 \otimes f_2 \otimes \cdots \otimes f_n).$$

Definition 2.3.8 An *algebra over an operad* $\mathscr{P}$ in a closed symmetric monoidal category $\mathbf{C}$ is an object $A \in \mathbf{C}$ equipped with an operadic morphism $\Phi : \mathscr{P} \to \text{End}(A)$.

Remark 2.3.9 Using tensor-hom adjunction we can see that this is equivalent to giving a sequence of products on A.

$$\Phi_r : \mathscr{P}(r) \otimes A^{\otimes r} \to A$$

In a recent preprint [25, Theorem A], Moreno-Fernández and Wierstra show that one can define a *coalgebra over an operad* in topological spaces. There is a similar, but not equivalent, notion in Vect given in [19, Subsection 5.2.17]. We summarise this below.

Definition 2.3.10 Let X be a pointed topological space. The *coendomorphism operad* $\mathrm{CoEnd}(X)$ has arity r component

$$\mathrm{CoEnd}(X)(r) := \mathrm{Map}_{\mathrm{Top}_*}(X, X^{\vee r})$$

For $r = 0$, set $\mathrm{CoEnd}(X)(0) = \mathrm{Map}_{\mathrm{Top}_*}(X, *) = *$. The operadic composition maps are defined by

$$\gamma : \mathrm{CoEnd}(X)(r) \otimes \mathrm{CoEnd}(X)(n_1) \otimes \cdots \otimes \mathrm{CoEnd}(X)(n_r) \to \mathrm{CoEnd}(X)(n_1 + \cdots + n_r)$$

$$(f, f_1, \cdots, f_n) \mapsto (f_1 \vee \cdots \vee f_n) \circ f$$

The symmetric group action permutes the wedge factors in the output.

Remark 2.3.11 Note that $\mathrm{CoEnd}(X)$ is naturally pointed. We will normally choose to ignore this extra structure, and will regard $\mathrm{CoEnd}(X)$ as unpointed for the rest of this report.

This immediately allows us to define a coalgebra as an algebra over the coendomorphism operad.

Definition 2.3.12 Let $\mathscr{P}$ be an (unpointed) operad in the category of topological spaces. A *$\mathscr{P}$-coalgebra* is a pointed space X along with an (unpointed) morphism of operads

$$\Delta : \mathscr{P} \to \mathrm{CoEnd}(X)$$

Remark 2.3.13 Using the product-mapping space adjunction we see that this is equivalent to giving a sequence of coproducts on the space X.

$$\Delta_r : \mathscr{P}(r) \times X \to X^{\vee r}$$

$$(\varphi, x) \mapsto \Delta(\varphi)(x)$$

We now have all the ingredients we need to proceed to the next topic where we will study algebras and coalgebras over one of the most important topological operads.

2.4 The little n-discs operad and the recognition principle

In this section we shall introduce the little n-discs operad $\mathbb{D}_n$ and briefly explain how one should think about it. Following that we shall see how loop spaces are algebras over the little discs operad and how they are basically unique among connected spaces in this. This uniqueness is called May's recognition principle. We shall end by showing that suspensions are coalgebras over $\mathbb{D}_n$.

Definition 2.4.1 Let D^n by the standard unit disc, defined as the space $\{(t_1, \dots, t_n) \in \mathbb{R}^n | t_1^2 + \cdots + t_n^2 \leq 1\}$ inside Euclidean space $\mathbb{R}^n$. A *little n-disc* is an affine embedding $c : D^n \to D^n$ of the form

$$c(t_1, \dots, t_n) = (a_1, \dots, a_n) + R \cdot (t_1, \dots, t_n)$$

for $(a_1, \dots, a_n) \in D^n$ and $R > 0$ such that $R^2 < 1 - (a_1^2 + \cdots + a_n^2)$. The *space of little n-discs* $\mathbb{D}_n(r)$ is the space of r-tuples $(c_1, \dots, c_r)$ whose terms are little discs such that $c_i^\circ \cap c_j^\circ = \emptyset$ for $i \neq j$. Each r-tuple can be identified with an element of the mapping space $\mathrm{Map}_{\mathrm{Top}_*}(\bigsqcup_{i=1}^n D^n, D^n)$ and so $\mathbb{D}_n(r)$ is equipped with the compact open topology.

The *little n-discs operad* $\mathbb{D}_n$ has underlying $\mathbb{S}$-module structure given by $\mathbb{D}_n(r)$ in arity r. It has a unit given by the identity homeomorphism in arity 1. We have a right $\mathbb{S}_n$-action given by $(c_1, \dots, c_r) * \sigma = (c_{\sigma(1)}, \dots, c_{\sigma(r)})$ for all $\sigma \in \mathbb{S}_n$. Finally the composition maps are

$$\begin{aligned} \gamma : \mathbb{D}_n(r) \times \mathbb{D}_n(n_1) \times \cdots \times \mathbb{D}_n(n_r) &\to \mathbb{D}_n(n_1 + \cdots + n_r) \\ ((c_1, \dots c_r), (c_{1,1}, \dots, c_{1,n_1}), \dots, (c_{1,1}, \dots, c_{1,n_r})) &\mapsto \\ (c_1 \circ c_{1,1}, \dots, c_1 \circ c_{1,n_1}, c_2 \circ c_{2,1}, \dots c_r \circ c_{r,n_r}). \end{aligned}$$

The above definition is long but its basic idea is quite simple. An element of arity r of the little n-discs operad is a disc with r little, non–overlapping discs drawn inside it. These discs are labelled using the natural numbers and the action of the symmetric group is to permute these labels. Figure 2.4 shows the action of $\mathbb{S}_3$ on an element of the little 2-discs operad.

Partial i^{th} composition is given by substituting a suitably scaled element of $D_2(n_i)$ where the little disc labelled i used to be. This is illustrated in Figure 2.4.

Remark 2.4.2 We say that a morphism of operads over an ambient model category **C** is a *weak homotopy equivalence* if the induced map of $\mathbb{S}$-modules is a weak equivalence in each arity. Two operads are weakly homotopy equivalent if there is a zig-zag of weak equivalences between them. An

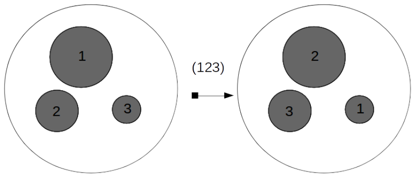


Figure 2.2: The action of (123) on an element of $D_2(3)$

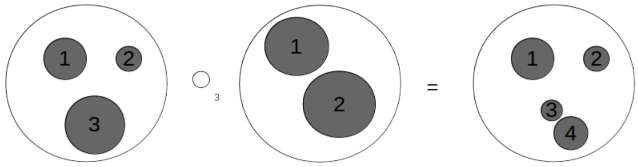


Figure 2.3: Partial composition in the operad of little discs

operad is said to be an E_n*-operad* if it is weakly homotopy equivalent to $\mathbb{D}_n$ and, for all $i \geq 1$, the right action of $\mathbb{S}_i$ on its arity i component is free. The homotopy theory of operads will be explored further in Chapter 3.

Example 2.4.3 The operad $\mathbb{D}_1$ is weakly homotopy equivalent to the operad Assoc. Thus Assoc is an E_1–operad.

The little n-discs operads all 'fit into' each other. We have a morphism $f : D^n \to D^{n+1}$, which sends D^n to the equatorial plane of D^{n+1}. We identify each little n-disc in D^n with a little $n+1$-disc in D^{n+1} centred in this plane and with the same radius. This produces a chain of inclusions

$$\mathbb{D}_1 \hookrightarrow \mathbb{D}_2 \hookrightarrow \mathbb{D}_3 \hookrightarrow \cdots \mathbb{D}_i \hookrightarrow \cdots \tag{2.1}$$

Example 2.4.4 We define $\mathbb{D}_\infty$ to be $\operatorname{colim}_n \mathbb{D}_n$, where the colimit is over the above chain seen as diagram. One can, with some pain, show that the discrete operad Com is weakly homotopy equivalent to $\mathbb{D}_\infty$.

The following example shows us what an algebra over the little discs operad looks like.

Example 2.4.5 Let X be a pointed topological space. The n-fold loop space $\Omega^n X$ has a natural $\mathbb{D}_n$–algebra structure. It is defined as follows. First, note that we can assume $\Omega^n X$ is of the form $\mathrm{Hom}_{\mathrm{Top}_*}(S^n, X)$ using the homeomorphism of Proposition 2.2.12. Then, for each $r \geq 1$, define an equivariant continuous map

$$\begin{aligned} \Phi'_n : \mathbb{D}_n(r) \times (\Omega^n X)^{\times r} &\to \Omega^n X \qquad (2.2) \\ (x, \alpha_1, \dots, \alpha_n) &\mapsto \overline{x(\alpha_1, \dots, \alpha_n)}. \end{aligned}$$

The notation $x(\alpha_1, \ldots, \alpha_n)$ needs to be explained. The element x of $\mathbb{D}_n(r)$ has the form $(f_1, f_2, \ldots f_r)$ where each $f_i : D^n \to D^n$ is a little n-disc. We define the map

$$x(\alpha_1, \ldots, \alpha_n) : D^n \to X$$

$$z \mapsto \begin{cases} \alpha_i(\pi(f_i^{-1}(z))) & \text{if } z \in f_i(D^n)^\circ \text{ for } 1 \leq i \geq r. \\ * & \text{otherwise.} \end{cases}$$

In the definition of this map, π the projection $D^n \to D^n/\partial D^n = S^n$. Finally, we observe that $x(\alpha_1, \ldots, \alpha_n)|_{\partial D^n}$ is the constant map $*$. Therefore $x(\alpha_1, \ldots, \alpha_n)$ restricts to a pointed map $\overline{x(\alpha_1, \ldots, \alpha_n)} : S^n \to X$, which exactly the map appearing in Equation 2.2. By adjunction, Φ'_n defines a map

$$\Phi_n : \mathbb{D}_n(r) \to \mathrm{Map}_{\mathrm{Top}_*}((\Omega^n X)^{\times r}, \Omega^n X)$$

which one can easily check is operadic morphism from the little n-discs operad to the coendomorphism operad of $\Omega^n X$.

A deep theorem of J. P. May states that, under appropriate conditions, every algebra of the little discs operad is of this form. This theorem is of great historical importance, motivating the introduction of operads themselves. It provides a necessary and sufficient condition for a space to be weakly homotopic to a loop space, or a *recognition principle*.

Theorem 2.4.6 (May's recognition principle [22]) *Let X be a connected topological space possessing a $\mathbb{D}_n$-algebra structure. Then it has the weak homotopy type of the n-fold loop space of some connected pointed space Y. Conversely every n-fold loop space is a $\mathbb{D}_n$-algebra as in Example 2.4.5.*

Remark 2.4.7 There are stronger versions of this statement available. In particular we can replace connected with *grouplike*. We can further say Y is $(n-1)$-connected space, meaning that all of its low dimensional homotopy groups ($\pi_i(X)$ for $i < n$) are trivial.

Remark 2.4.8 All that we have said in the this section, including May's recognition principle, can rephrased in terms of the *little n-cubes operad.* This is essentially the same as the little n-discs operad, except that the word 'n-disc[s]' is replaced everywhere with 'n-cube[s]'. As one would expect, the resulting operad can be shown to be E_n. Using the little n-cubes operad instead of the little n-discs operad can occasionally greatly simplify calculations. We shall encounter one such situation in Chapter 3 when we wish to show that the little n-discs operad is a cellular E_n operad.

The maps we defined in Equation 2.2 descend to the (singular) homology and induce various operations on it and the cohomology. For further details, we refer you to [6]. We will finish this chapter with the Moreno–Fernández–Wierstra construction of the coalgebras over the little discs operad. This construction is the Eckmann-Hilton dual of Example 2.4.5 and therefore takes place over n-fold suspensions.

Example 2.4.9 [25] The n-sphere S^n possesses a $\mathbb{D}_n$–coalgebra structure. It is constructed as follows. As in Example 2.4.5, for each $r \geq 1$, we define a map

$$P_r' : \mathbb{D}_n(r) \times D^n \to (\bigsqcup_{i=1}^{r} D^n) \sqcup \{*\}$$

$$(x, y) \mapsto \begin{cases} * & \text{if } y \notin (f_i(D^n))^\circ \text{ for } 1 \leq i \leq r. \\ f_i^{-1}(y) & \text{otherwise.} \end{cases}$$

As in Example 2.4.5, x is identified with $(f_1, \ldots f_r)$ where each f_i is a little n-disc. Note that we have a natural projection $\kappa : (\bigsqcup_{i=1}^{r} D^n) \sqcup \{*\} \to (S^n)^{\vee r}$ given by identifying the boundaries of all of the D^n with $*$. Therefore, we have a map

$$\mathbb{D}_n(r) \times D^n \xrightarrow{P_r'} (\bigsqcup_{i=1}^{r} D^n) \sqcup \{*\} \xrightarrow{\kappa} (S^n)^{\vee r}.$$

Observe that $\kappa \circ P_r'|_{':\mathbb{D}_n(r)\times\partial D^n}$ is the constant map $*$. Therefore $\kappa \circ P_r'$ factors as

$$\mathbb{D}_n(r) \times D^n \xrightarrow{\text{id}\times\pi} \mathbb{D}_n(r) \times S^n \xrightarrow{\Delta_n'} (S^n)^{\vee r}.$$

where π is once again the projection $D^n \to D^n/\partial D^n$. One can easily verify that this map $\Delta_n' : \mathbb{D}_n(r) \times S^n \to (S^n)^{\vee r}$ is continuous and equivariant. Therefore, by the Cartesian product-mapping space adjunction, we have a continuous equivariant map

$$\Delta_n : \mathbb{D}_n(r) \to \text{Map}_{\text{Top}_*}(S^n, (S^n)^{\vee r})$$

and one can easily check that this defines an operadic morphism to the coendomorphism operad of S^n.

Example 2.4.9 generalises easily to the reduced suspension of any space. We will phrase this result as a theorem.

Theorem 2.4.10 *[25, Theorem 2.1] Let $\Sigma^n X$ be the n-fold suspension of a pointed space X. Then there is a natural map of operads*

$$\Delta : \mathbb{D}_n \to \text{CoEnd}(\Sigma^n X)$$

which encodes the homotopy coassociativity and homotopy cocommutativity of the pinch map. Otherwise said, n-fold suspensions are coalgebras over the little n-discs operad.

Proof Using Remark 2.2.13 we have that

$$\Sigma^n(Y \vee Z) = (Y \vee Z) \wedge S^n = (Y \wedge S^n) \vee (Z \wedge S^n) = \Sigma^n Y \vee \Sigma^n Z$$

where the second equality follows from Proposition 2.2.9.
We define the map $\Sigma^n X \to (\Sigma^n X)^{\vee r}$ as the composition

$$\Sigma^n X \cong X \wedge S^n \xrightarrow{id_x \wedge \Delta_r(x)} (X \wedge (S^n)^{\vee r}) \cong (X \wedge S^n)^{\vee r} \cong (\Sigma^n X)^{\vee r}.$$

One can easily check that these maps define an operadic morphism. □

Chapter 3

Simplicial sets

Simplicial sets were first introduced in 1950 by Samuel Eilenberg and J. A. Zilber [11], as a way of combinatorially storing and manipulating the data of sufficiently well-behaved topological spaces. This chapter provides a classical treatment of them. It also includes two important special topics - Kan's Ex^{∞} functor and the Barratt-Eccles operad. Simplicial sets, and more generally simplicial objects, have too many uses to possibly enumerate. Remark 3.5.2 gives one example, they can be used to construct the classifying space (up to homotopy) of any sufficiently well-behaved topological group. Simplicial objects also play an important role in the classical proof of May's recognition principle [22]. More recently, simplicial sets have been used to define the basic notion of an ∞-categories in higher algebra [20].

In Section 3.1, we shall define simplicial sets and study their structure as combinatorial objects. In Section 3.2, we shall see how the category of simplicial sets is connected to topological spaces via a Quillen equivalence. Following that, in Section 3.3, we shall study the analogues in simplicial sets of the special constructions introduced in Section 2.2; the wedge and smash products, and suspensions and loop spaces. In Section 3.4, we shall introduce Kan's Ex^{∞} functor; a completely combinatorial functor that computes fibrant replacements in the category of simplicial sets. Finally, we shall conclude this chapter in Section 3.5 by studying the Barratt-Eccles E_n-operad, the simplicial analogue of the little n-discs operad. Some of this discussion will feed into the next chapter, where we discuss the model structure on operads.

3.1 Simplicial sets

In this section we will introduce the simplex category and define simplicial sets as presheaves over it. We will also study some essential basic examples. Simplicial sets are an important tool, analogous to CW complexes but more powerful, for capturing information about topological spaces in a completely combinatorial way. The material in this section is quite classical and there are many good references available. We have chosen to mainly follow [13, Chapter 1] and [14].

Definition 3.1.1 The *simplex category* $\triangle$ is the category whose objects are the nonempty finite ordinals, traditionally denoted as the totally ordered sets $[n] = \{0, 1, \ldots, n\}$. The morphisms $\mathrm{Hom}_{\triangle}([n], [m])$ of $\triangle$ consist of the order-preserving (or *increasing*) functions $[n] \to [m]$.

Every morphism of $\triangle$ admits a unique factorisation as an increasing surjection followed by an increasing injection. This observation motivates us to study the following two important classes of increasing functions.

Definition 3.1.2 We define $\partial^i : [n-1] \to [n]$ by

$$\partial^i(x) = \begin{cases} x & \text{if } 0 \leq x < i \\ x+1 & \text{if } i \leq x \leq n-1 \end{cases} \qquad \text{for } 0 \leq i \leq n. \tag{3.1}$$

We define $\sigma^j : [n+1] \to [n]$ by

$$\sigma^j(x) = \begin{cases} x & \text{if } 0 \leq x \leq j \\ x-1 & \text{if } j < x \leq n \end{cases} \qquad \text{for } 0 \leq j \leq n. \tag{3.2}$$

Remark 3.1.3 In theory we should denote these maps by ∂^i_n and σ^i_n respectively. In practise it is almost always clear from context what the domain and codomain are and therefore it is standard to omit the index n.

These maps generate all the morphisms of $\triangle$ and satisfy the following easily verifiable *cosimplicial relations*.

$$\begin{aligned} \partial^j \partial^i &= \partial^i \partial^{j-1} \text{ if } i < j \\ \sigma^j \partial^i &= \partial^i \sigma^{j-1} \text{ if } i < j \\ \sigma^j \partial^i &= \mathrm{id} \ \text{ if } i = j, j+1 \\ \sigma^j \partial^i &= \partial^{i-1} \sigma^j \text{ if } i > j+1 \\ \sigma^j \sigma^i &= \sigma^{i-1} \sigma^j \text{ if } i \geq j \end{aligned} \tag{3.3}$$

Definition 3.1.4 A *simplicial set* is a (contravariant) functor $X_\bullet : \triangle^{op} \to \mathrm{Set}$.

Remark 3.1.5 Dually, we have the notion of a *cosimplicial set* which is a covariant functor $X_\bullet : \triangle \to \mathrm{Set}$. More generally, for an arbitrary category C any (contravariant) functor $X_\bullet : \triangle^{op} \to C$ is referred to as a *simplicial object*. In particular, we shall later encounter the concept of a simplicial group.

More combinatorially, a simplicial set $X_\bullet$ consists of the following data:

- For each $n \in \mathbb{N}_0$, a set X_n, the elements of which are called *n-simplices.* 0-simplices are usually called *points*, 1-simplices *edges* and 2-simplices *faces.*
- *face maps* $d_i : X_n \to X_{n-1}$ for $0 \leq i \leq n$.
- *degeneracy maps* $s_j : X_n \to X_{n+1}$ for $0 \leq j \leq n$.

The face and degeneracy maps satisfy the *simplicial relations*, which are the cosimplicial relations with the order of composition reversed. Explicitly, they are

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \text{ if } i < j. \\ d_i s_j &= s_{j-1} d_i \text{ if } i < j. \\ d_i s_j &= \text{id} \text{ if } i = j, j+1. \\ d_i s_j &= s_j d_{i-1} \text{ if } i > j+1. \\ s_i s_j &= s_j s_{i-1} \text{ if } i \geq j. \end{aligned} \tag{3.4}$$

We refer to simplices that are in the image of any degeneracy map as *degenerate.*

Example 3.1.6 A very common way for simplicial sets to arise ‘in nature’ is as the *nerve* of a small category $\mathcal{J}$, denoted $\mathcal{N}(\mathcal{J})$. This is given by

$$\mathcal{N}(\mathcal{J})_n = \text{Hom}_{\text{Cat}}([n], \mathcal{J})$$

where $[n]$ is the poset (regarded as a small category with inclusion morphisms) $\{0, 1, \ldots, n\}$, and with the natural face and degeneracy maps. More explicitly $\mathcal{N}(\mathcal{J})_n$ consists of n-tuples of composable morphisms

$$A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_n$$

in $\mathcal{J}$ with face maps d_i given by composing the morphisms $A_{i-1} \to A_i$ and $A_i \to A_{i+1}$ to $A_{i-1} \to A_{i+1}$ and degeneracy maps s_i given by adding an identity map at A_i.

Definition 3.1.7 The category $\text{Set}_\triangle$ consists of all simplicial sets with the morphisms between them being natural transformations of functors.

Remark 3.1.8 More prosaically, a morphism of simplicial sets from $X_\bullet$ to $Y_\bullet$ can also be described as a collection of functions $f_n : X_n \to Y_n$, which commutes with the face and degeneracy maps.

Definition 3.1.9 A *simplicial subset* of $X_\bullet$ is a simplicial set $Y_\bullet$ such that $Y_n \subseteq X_n$. The face and degeneracy maps on $Y_\bullet$ are inherited from $X_\bullet$, and in particular, $Y_\bullet$ is closed under them.

There are two related kinds of simplicial sets that have properties that make them of particular interest.

Example 3.1.10 The *standard n-simplex* $\triangle^n_\bullet$ is defined to be $\mathrm{Hom}_\triangle([k],[n])$. with degeneracy and face maps induced by the functorality of $\mathrm{Hom}(-,[n])$. By Yoneda's lemma, for any simplicial set X we have $\mathrm{Hom}_{\mathrm{Set}_\triangle}(\triangle^n, X) \cong X_n$.

Consider $v_n = \mathrm{id}_{[n]} \in \triangle^n_n$. The *boundary* of the standard simplex $\partial\triangle^n_\bullet$ is the smallest simplicial subset of $\triangle^n_\bullet$ which contains $d_i v_n$, the i^{th} *face*, for $0 \leq i \leq n$. We interpret this combinatorially as

$$\partial\triangle^n_i = \{f : [i] \to [n] | f \in \triangle^n_i \text{ and } f \text{ is not surjective}\}.$$

Later, when we define the geometric realization of a simplex, we shall see that the boundary of a simplex is realized as the boundary of the corresponding geometric simplex.

Remark 3.1.11 We can define a natural cosimplicial object $\triangle^\bullet : \triangle \to \mathrm{Set}_\triangle$ in the category of simplicial sets as follows.

- It has dimension n component $\triangle^n$ equal to $\triangle^n_\bullet$.
- The i^{th} coface map (in dimension k) $\partial^i_{\triangle^\bullet,k} : \triangle^n_k \to \triangle^{n+1}_k$ is defined by

$$\partial^i_{\triangle^\bullet,k} := \mathrm{Hom}_\triangle([k], \partial^i).$$

- The i^{th} codegeneracy map (in dimension k) $\sigma^i_{\triangle^\bullet,k} : \triangle^n_k \to \triangle^{n-1}_k$ is defined by

$$\sigma^i_{\triangle^\bullet,k} := \mathrm{Hom}_\triangle([k], \sigma^i).$$

Some of this notation may be confusing, so it is worth remarking that the ∂^i and σ^i occurring within the expressions $\mathrm{Hom}_\triangle([k], \partial^i)$ and $\mathrm{Hom}_\triangle([k], \sigma^i)$ are morphisms in the simplex category $\triangle$ and entirely distinct from $\partial^i_{\triangle^\bullet}$ and $\sigma^i_{\triangle^\bullet}$.

There is an analogue of Definition 2.2.1 in category of simplicial sets (These are both examples of a more general phenomenon called an *internal hom* see Definition 4.1.1 for further details), called the *simplicial mapping space*. It is defined as follows.

Definition 3.1.12 Let X and Y be simplicial sets. The *simplicial mapping space* $\mathrm{Map}_{\mathrm{Set}_\triangle}(X, Y)$ is a simplicial set with dimension n component given by

$$\mathrm{Map}_{\mathrm{Set}_\triangle}(X, Y)_n = \mathrm{Hom}_{\mathrm{Set}_\triangle}(X \times \triangle^n, Y)$$

and with face maps

$$d_i : \mathrm{Map}_{\mathrm{Set}_\triangle}(X, Y)_n \to \mathrm{Map}_{\mathrm{Set}_\triangle}(X, Y)_{n-1}$$

$$f \mapsto (\text{id} \times \partial^i_{\triangle^\bullet}) \circ f$$

and with degeneracy maps

$$s_i : \text{Map}_{\text{Set}_\triangle}(X,Y)_n \to \text{Map}_{\text{Set}_\triangle}(X,Y)_{n+1}$$

$$f \mapsto (\text{id} \times \sigma^i_{\triangle^\bullet}) \circ f.$$

The final topic of this section is skeletons of simplicial sets.

Definition 3.1.13 Let X be a simplicial set and let n be a non-negative integer. The *n-skeleton* of X, written $\text{sk}_n X$ is the smallest simplicial subset of X that contains all the non-degenerate simplices of dimension $\leq n$.

Remark 3.1.14 This is the same idea as that of n-skeletons of CW-complexes. In both cases we simply forget the higher dimensional cells. The only difference that $(\text{sk}_n X)_k$ will not be empty for $k > n$. Instead it will contain the degeneracies of the lower dimensional simplices.

Remark 3.1.15 A standard method of proof in simplicial set theory is to prove a property via induction on the n-skeleton.

3.2 The relationship between $\text{Set}_\triangle$ and Top

As we have mentioned several times, there is a strong relationship between simplicial sets and topological spaces. To be precise, there is a model category structure on $\text{Set}_\triangle$ which is Quillen equivalent to the standard model structure in Top . We describe the adjoint pair of functors between these categories first, as it provides the easiest way to describe the weak equivalences in $\text{Set}_\triangle$. Our treatment broadly follows [13, Chapters 1 & 2].

Definition 3.2.1 The *geometric n-simplex* is defined to be

$$|\triangle^n| = \{(t_0, \dots t_n) \in (\mathbb{R}_{\geq 0})^{n+1} : t_0 + \cdots + t_n = 1\}.$$

The collection of geometric n-simplex $|\triangle^\bullet|$ forms a cosimplicial object in the category of topological spaces when equipped with coface maps via

$$\partial^i : |\triangle^n| \to |\triangle^{n+1}|$$

$$(t_0, \dots t_n) \mapsto (t_0, \dots, t_i, 0, t_{i+1}, \dots t_n),$$

and codegeneracy maps via

$$\sigma^i : |\triangle^n| \to |\triangle^{n-1}|$$

$$(t_0, \dots t_n) \mapsto (t_0, \dots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \dots, t_n).$$

Definition 3.2.2 Let $X_\bullet$ be a simplicial set. Its *geometric realisation* is the topological space

$$|X_\bullet| := \Big(\bigsqcup_{n \geq 0} X_n \times |\triangle^n| \Big) / \sim$$

where the equivalence relation $\sim$ is given by

$$(d_i(x), t) \sim (x, \partial^i(t)), \qquad (s_j(x), t) \sim (x, \sigma^j(t)).$$

Example 3.2.3 The geometrical realization of the standard n-simplex is the geometric n-simplex.

The geometric realisation is functorial and gives us our desired functor $F : \mathrm{Set}_\triangle \to \mathrm{Top}$. The next construction is also functorial and provides us with the adjoint functor in the opposite direction.

Definition 3.2.4 Let Y be a topological space. Its *singular simplicial set* is given by $S_\bullet(Y) := \mathrm{Hom}_{\mathrm{Top}}(|\triangle^\bullet|, Y)$ with face and degeneracy maps induced by the cosimplicial structure on $|\triangle^\bullet|$.

Proposition 3.2.5 *[14] The singular simplicial set and the geometric realisation functors form an adjunction, ie.* $S_\bullet : \mathrm{Top} \rightleftarrows \mathrm{Set}_\triangle : |\ \ |$

Remark 3.2.6 The proof of this is omitted, but very straightforward.

The model category structure on $\mathrm{Set}_\triangle$ is chosen precisely to make this adjunction Quillen.

Definition 3.2.7 Let $n \geq 1$ and $0 \leq k \leq n$. The k^{th} horn $(\Lambda^n_k)_\bullet$ is the smallest simplicial subset of $\triangle^n_\bullet$ that contains the faces $d_i v_n$ for $i \neq k$. Otherwise said, the k^{th} horn is the boundary of the n-simplex missing the k^{th} face.

Definition 3.2.8 A *Kan fibration* is a simplicial morphism $p : X \to Y$ that has the right lifting property with respect to all inclusions $\Lambda^n_k \subset \triangle^n$. In other words the dotted lift exists in all diagrams of the form

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ \triangle^n & \longrightarrow & Y. \end{array}$$

One can check quite easily that the geometric realisation of a Kan fibration is a Serre fibration. We define Kan complexes to be the objects with a Kan fibration as their terminal morphism. As these will play an important role in this report, we will write this out more formally.

Definition 3.2.9 A *Kan complex* is a simplicial set $X_\bullet$ such that, for every $n \in \mathbb{N}$ and $n \geq k \geq 0$, the dotted arrow exists in every diagram of the form

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \triangle^n. & & \end{array}$$

This lifting property is also called the *horn filling condition*.

Theorem 3.2.10 *[13, Theorem 11.3] The category* $\mathrm{Set}_\triangle$ *admits a cofibrantly generated model category structure, called the Kan–Quillen structure, where*

- *The weak equivalences are the simplicial morphisms that induce weak homotopy equivalences on the level of geometric realisations.*
- *The fibrations are the Kan fibrations.*
- *The cofibrations are the levelwise injective simplicial morphisms.*

In the Kan–Quillen model structure every simplicial set is cofibrant and the fibrant objects are the Kan complexes. Moreover, the Quillen adjunction between $S_\bullet$ and $|\;|$ becomes a Quillen equivalence.

Theorem 3.2.11 *[13, Theorem 11.4] When* $\mathrm{Set}_\triangle$ *and* Top *are equipped with their respective Quillen model structures, the adjunction between the geometric realisation and singular simplicial set functors* $S_\bullet : \mathrm{Top} \rightleftarrows \mathrm{Set}_\triangle : |\;\;|$ *is a Quillen equivalence.*

3.3 Construction on simplicial sets

In this section, we will describe the simplical analogues of the classical topological constructions of Section 2.2. We start by defining the category of pointed simplicial sets. We then define the simplicial wedge and smash products. Afterwards we look at the loop spaces and the reduced suspension. Our treatment here is similar to that of Curtis in [7].

Definition 3.3.1 A *base point* $*$ of a simplicial set X is the simplicial subset of X consisting of a vertex and all of its degeneracies. By abuse of notation, $*_n$ is usually abbreviated to $*$.

Remark 3.3.2 A *pointed simplicial set* is a pair $(X, *)$ consisting of a simplical set and a base point $*$. A morphism of pointed simplicial sets is simply a simplicial morphism that preserves base points.

Definition 3.3.3 Let X be a simplicial set and Y be a simplicial subset. The *simplicial quotient* is defined by $(X/Y)_n := X_n/Y_n$.

Example 3.3.4 A very important example of a pointed simplicial set is the n-sphere S^n. This is defined as the quotient $\triangle^n/\partial\triangle^n$. It contains two nondegenerate simplices, one 0-simplex corresponding to $\partial\triangle^n$ which we take to be $*$ and one n-simplex corresponding to $\mathrm{id}_{[n]}$.

The most naïve way to combine two simplicial sets is the Cartesian product. All of our later binary operations will be defined in terms of it.

Definition 3.3.5 Let X and Y be two simplicial sets. The *Cartesian product* $X \times Y$ is defined by $(X \times Y)_n = X_n \times Y_n$ with the face and degeneracy maps given by $d_i(x,y) = (d_i x, d_i y)$ and $s_i(x,y) = (s_i x, s_i y)$.

Remark 3.3.6 The Cartesian product is the category-theoretical product in the category of simplicial sets. In other words, the following diagram is a pullback

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_1} & X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{q} & *. \end{array}$$

where; the π_i are the projection maps, $*$ is the terminal object in the category of simplicial sets (ie. the simplicial set with one non-degenerate simplex in dimension 0) and p, q are terminal morphisms.

Definition 3.3.7 The *wedge product* $X \vee Y$ of two simplicial sets is the simplicial subset of $X \times Y$ consisting of the union of $X \times *$ and $* \times Y$. The *smash product* $X \wedge Y$ of two simplicial sets is defined to be $(X \times Y)/(X \vee Y)$ These operations on simplicial sets respectively correspond to the wedge sum and smash product on topological spaces. One can prove that the smash product is the category-theoretical product for the category of pointed simplicial sets.

Definition 3.3.8 Let $(X, *)$ be a pointed simplicial set. The *loop space* ΩX of X is the simplicial set

$$(\Omega X)_n = \{x \in X_{n+1} : d_1 d_2 \ldots d_{n+1}(x) = * \text{ and } d_0(x) = *\}.$$

For each $0 \leq i \leq n$, the face map $d_i^{\Omega X}$ is the restriction of the face map d_{i+1}^X of X_{n+1} to $(\Omega X)_n$. Similarly the degeneracy map $s_i^{\Omega X}$ is the restriction of the face map s_{i+1}^X of X_{n+1}.

Definition 3.3.9 Let X be a simplicial set. The *reduced cone* is the simplicial set

$$(CX)_n = \{*\} \cup \{(x,q) : x \in X_{n-q} - \{*\} \text{ and } 0 \leq q \leq n\}/$$

The face and degeneracy maps are given by

$$d_i(x,q) = \begin{cases} (x, q-1), & \text{if } 0 \leq i < q \\ (d_{i-q}x, q), & \text{if } q \leq i \leq n \end{cases}$$

$$s_i(x,q) = \begin{cases} (x, q+1), & \text{if } 0 \le i < q \\ (s_{i-q}x, q), & \text{if } q \le i \le n. \end{cases}$$

Definition 3.3.10 Observe that there is an embedding $X \hookrightarrow CX$ given combinatorially by $x \mapsto (x, 0)$. The *(reduced) suspension* ΣX of the simplicial set X is the pointed simplicial set CX/X.

As one would expect, all the relations between these objects that we have seen in the category of topological spaces hold in the simplicial world. For example:

Proposition 3.3.11 *The n-fold suspension $\Sigma^n X$ of a simplicial set X is isomorphic to $X \wedge S^n$.*

In order to show this, we will first prove two lemmata.

Lemma 3.3.12 *The suspension ΣX of a simplicial set X is isomorphic to $X \wedge S^1$.*

Proof The 1-sphere S^1 has two non-degenerate simplices, $*$ in dimension 0 and $\sigma \in S^1_1$. It follows that in dimension n, S^1_n is the set

$$\{*\} \cup \{s_{(k)}(\sigma) : 0 \le k \le n-1\}$$

where $s_{(k)} = s_{n-1}s_{n-2}\cdots\widehat{s_k}\cdots s_0$. We are using the notation $\widehat{x}$ to mean that we are omitting x. From this we can concretely describe $X \wedge S^1$ as

$$(X \wedge S^1)_n = \{*\} \cup \{(x, s_{(q)}(\sigma)) : x \in X_n - \{*\} \text{ and } 0 \le q \le n-1\} \quad (3.5)$$

$$d_i(x, s_{(q)}(\sigma)) = \begin{cases} (d_i x, s_{(q-1)}(\sigma)), & \text{if } 0 \le i \le q \\ (d_i x, s_{(q)}(\sigma)), & \text{if } q < i < n \end{cases}$$

$$s_i(x, s_{(q)}(\sigma)) = \begin{cases} (s_i x, s_{(q+1)}(\sigma)), & \text{if } 0 \le i \le q \\ (s_i x, s_{(q)}(\sigma)), & \text{if } q < i < n \end{cases}$$

Now we define a pointed simplicial morphism $f : \Sigma X \to X \wedge S^1$ as

$$f : (x, q) \to (s_{q-1}\cdots s_0(x), s_{(q-1)}(\sigma))$$

where we are using the description of the reduced suspension descending from that of the cone. This is defined on the whole domain because $(x, 0)$ goes to $*$ in the quotient CX/X. Checking that this is indeed a simplicial morphism is straightforward task using the standard s_i and d_i identities. We observe that it is invertible and thus induces an isomorphism of simplicial sets. □

Lemma 3.3.13 *The n-sphere S^n is isomorphic to $(S^1)^{\wedge n}$.*

Proof As in the proof of Lemma 3.3.12 we observe the elements of $S^n = \triangle^n/\partial\triangle^n$ consist of a 0-simplex $*$, an m-simplex σ_m and their degeneracies. The smash product $(S^1)^{\wedge n}$ can be written

$$(S^1)^{\wedge n}_m = \{*\} \cup \{(s_{(q_1)}(\sigma_1), s_{(q_2)}(\sigma_1), \dots s_{(q_n)}(\sigma_1)) : m > q_i \geq 0\} \tag{3.6}$$

and is equipped with face and degeneracy maps in the obvious way.

From standard simplicial identities, one can check that for every permutation $\tau \in \mathbb{S}_n$

$$(s_{(\tau(0))}(\sigma_1), \dots, s_{(\tau(n-1))}(\sigma_1))$$

is the same as

$$(s_{(0)}(\sigma_1), \dots, s_{(n-1)}(\sigma_1)).$$

In other words, $(S^1)^{\wedge n}_n$ has only one nondegenerate simplex. Therefore one can define a pointed simplicial map $f : S^n \to (S^1)^{\wedge n}$ by

$$\sigma_n \mapsto (s_{(0)}(\sigma_1), \dots, s_{(n-1)}(\sigma_1))$$

Observe that $(S^1)^{\wedge n}$ has no nondegenerate simplices in dimension $m > n$. If it did, this simplex would have to be of the form

$$(s_{(q_1)}(\sigma_1), \dots, s_{(q_m)}(\sigma_1)). \tag{3.7}$$

and there would exist least one $0 \geq j < m$ such that s_j is a component of all the $s_{(q_i)}$. All the simplices of dimension $m < n$ are also of the form 3.7, which means they are degeneracies of $f(\sigma_n)$. Therefore, f is surjective. Since it is clearly injective, we deduce that f is an isomorphism. □

Proof (Proposition 3.3.11) By Lemma 3.3.12 we know that $\Sigma X = X \wedge S^1$. Thus for general n, $\Sigma^n X = X \wedge (S^1)^{\wedge n}$. Whence by Lemma 3.3.13 we know $\Sigma^n X \cong X \wedge S^n$. □

3.4 Kan's Ex^∞ functor

One of the basic facts about a model category is that all objects admit a functorial fibrant and cofibrant replacements. Calculating these is extremely useful as it allows easy passage to the homotopy category. In the category $\mathrm{Set}_\triangle$ equipped with Quillen model structure, every object is cofibrant and so the cofibrant replacement functor is trivial.

The question of fibrant replacements is resolved via Kan's Ex^∞ functor which computes them via the process of barycentric subdivision. We shall be broadly following Chapter III of [13].

$$0 \longrightarrow (0,1) \longleftarrow 1$$

Figure 3.1: $\mathrm{sd}\,\triangle^1$

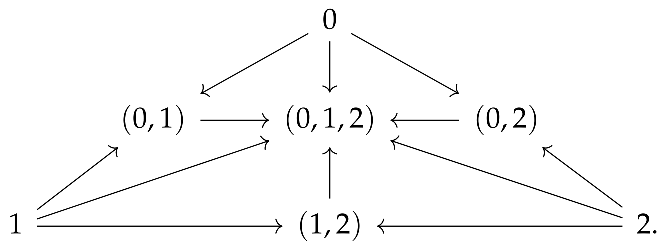


Figure 3.2: $\mathrm{sd}\,\triangle^2$

Definition 3.4.1 Observe that the nondegenerate simplices of $\triangle^n$ are exactly the increasing injections $[m] \to [n]$ with $0 \leq m \leq n$. These are in one-to-one correspondence with the subsets of $\{0, 1, \ldots, n\}$ of cardinality $m+1$ and thus form a poset under inclusion which we denote $P\triangle^n$. We define the *simplical subdivision* of $\triangle^n$ to be

$$\mathrm{sd}\,\triangle^n := \mathcal{N}(P\triangle^n)$$

where $\mathcal{N}$ is the nerve of the poset (regarded as a small category with morphisms given by inclusions).

Visually the subdivisions of the first two standard simplices are shown in Figures 3.1 and 3.2. The arrow notation

$$a \xrightarrow{f} b$$

simply means that $d_0(f) = a$ and $d_1(f) = b$. The leading way that they have been drawn, illustrates the following lemma.

Lemma 3.4.2 *[13, Lemma III.4.1] On the level of geometric realizations, there is a homeomorphism* $f : |\,\mathrm{sd}\,\triangle^n| \xrightarrow{\sim} |\triangle^n|$.

The notion of subdivision can be extended to any simplicial set, not just the standard simplices. This extension makes use of the notion of a *simplex category*, which we shall introduce next.

Definition 3.4.3 The *simplex category* $\triangle \downarrow X$ of a simplicial set X, has for objects all simplicial maps $\sigma : \triangle^n \to X$ and has for morphisms, the commutative diagrams of the form

$$\begin{array}{ccc} \triangle^n & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle \theta^*} & \nearrow{\scriptstyle \tau} & \\ \triangle^m & & \end{array}$$

where θ^* is induced by a unique ordinal map $\theta : [m] \to [n]$.

Definition 3.4.4 Let X be a simplicial set. The *subdivision* $\operatorname{sd} X$ of X is defined to be the simplicial set

$$\operatorname{sd} X = \lim_{\triangle^n \to X} \operatorname{sd} \triangle^n$$

with the limit indexed by the simplex category of X.

Remark 3.4.5 This construction is functorial.

Definition 3.4.6 Let X be a simplicial set. There is a natural map $\nu_{\triangle^n}$: $\operatorname{sd} \triangle^n \to \triangle^n$ induced by the map of posets $P\triangle^n \to [n]$ given by

$$[v_0, v_1, \ldots, v_k] \mapsto v_k.$$

The *last vertex map* $\nu_X : \operatorname{sd} X \to X$ is

$$\nu_X = \lim_{\triangle^n \to X} \nu_{\triangle^n}$$

with the limit indexed by the simplex category of X.

Lemma 3.4.7 *Let X and Y be simplicial sets and $f : X \to Y$ be a simplicial map. Then the following diagram is commutative.*

$$\begin{array}{ccc} \operatorname{sd} X & \xrightarrow{\nu_X} & X \\ \big\downarrow{\scriptstyle \operatorname{sd} f} & & \big\downarrow{\scriptstyle f} \\ \operatorname{sd} Y & \xrightarrow{\nu_Y} & Y. \end{array} \tag{3.8}$$

Proof This follows from the functorality of $\lim_{\triangle^n \to X}$. □

We define the Ex functor to be the right adjoint of the sd functor.

Definition 3.4.8 For any simplicial set X we define

$$\operatorname{Ex}(X)_n := \operatorname{Hom}_{\mathrm{Set}_\triangle}(\operatorname{sd} \triangle^n, X)$$

Definition 3.4.9 We have a morphism $\mu_X : X \to \operatorname{Ex}(X)$ which is adjoint to the last vertex map. Thus we obtain a diagram

$$X \longrightarrow \operatorname{Ex}(X) \longrightarrow \operatorname{Ex}^2(X) \longrightarrow \cdots \tag{3.9}$$

The colimit of this diagram is denoted $\operatorname{Ex}^\infty(X)$.

Example 3.4.10 If X is the one point simplicial set, then $\operatorname{Ex}^\infty(X)$ is isomorphic to X. Too see this, observe that the one point simplicial set $*$ is the terminal object in the category of simplicial sets. Therefore $(\operatorname{Ex} *)_n = \operatorname{Hom}_{\mathrm{Set}_\triangle}(\operatorname{sd} \triangle^n, *) = *_n$. The face and degeneracy maps are obviously also uniquely determined. Thus we can conclude that $\mathrm{Ex}\,* = *$ and thence by induction that $\mathrm{Ex}^n * = *$. The map $\mathrm{Ex}^n * \to \mathrm{Ex}^{n+1} *$ induced by the last vertex map must be the identity by the terminality of $*$, and the limit of the diagram

$$* \longrightarrow * \longrightarrow * \longrightarrow \cdots$$

is $*$.

The following theorem lists some useful properties of the Ex^∞ functor.

Theorem 3.4.11 (Properties of the Ex^∞ functor) *[13, Theorem 4.8] Let X be a simplicial set. Then:*

1. Ex^∞*(X) is a Kan complex.*
2. *The canonical map $\eta_X : X \to \mathrm{Ex}^\infty(X)$ is an acyclic cofibration.*
3. Ex^∞ *preserves fibrations.*
4. Ex^∞ *preserves finite limits.*
5. *There is a homotopy equivalence between $\mathrm{Ex}^\infty(X)$ and $\mathrm{Ex}^\infty(\mathrm{Ex}^\infty(X))$.*

Proof 1. It suffices to prove the following statement; for any diagram of the form

$$\varphi : \Lambda^n_k \to \mathrm{Ex}(X)$$

there is a unique extension

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\varphi} & \mathrm{Ex}(X) \\ \downarrow & & \downarrow{\scriptstyle j_{\mathrm{Ex}(X)}} \\ \triangle^n & \dashrightarrow & \mathrm{Ex}^2(X) \end{array}$$

To prove this, observe first that φ factorises as

$$\Lambda^n_k \xrightarrow{f} \mathrm{Ex}(\mathrm{sd}\,\Lambda^n_k) \xrightarrow{\mathrm{Ex}(\varphi)} \mathrm{Ex}(X)$$

Therefore we can obtain our desired extension via proving the existence of the dotted map in the following diagram.

$$\begin{array}{ccccc} \Lambda^n_k & \xrightarrow{f} & \mathrm{Ex}(\mathrm{sd}\,\Lambda^n_k) & \xrightarrow{\mathrm{Ex}(\varphi)} & \mathrm{Ex}(X) \\ \downarrow & & \downarrow{\scriptstyle j_{\mathrm{Ex}(\mathrm{sd}\,\Lambda^n_k)}} & & \downarrow{\scriptstyle j_{\mathrm{Ex}(X)}} \\ \triangle^n & \dashrightarrow & \mathrm{Ex}^2(\mathrm{sd}\,\Lambda^n_k) & \xrightarrow{\mathrm{Ex}^2(\varphi)} & \mathrm{Ex}^2(X) \end{array}$$

The composite map $j_{\mathrm{Ex}(\mathrm{sd}\,\Lambda^n_k)} \circ f$ is identical to the map

$$\Lambda^n_k \xrightarrow{j_{\Lambda^n_k}} \mathrm{Ex}(\Lambda^n_k) \xrightarrow{\mathrm{Ex}(f)} \mathrm{Ex}^2(\mathrm{sd}\,\Lambda^n_k)$$

Thus, using the adjunction between Ex and sd, the problem reduces to finding the dotted map in the next diagram.

$$\begin{array}{ccc} \operatorname{sd}\Lambda^n_k & \longrightarrow & \Lambda^n_k \\ \big\downarrow & & \big\downarrow f \\ \operatorname{sd}\triangle^n & \dashrightarrow & \operatorname{Ex}(\operatorname{sd}\Lambda^n_k) \end{array}$$

We now explicitly construct this map, by defining as follows, for all $\sigma = (\sigma_0, \ldots, \sigma_m) \in (\operatorname{sd}\Lambda^n_k)_m$ with $\sigma_i \in \triangle^n_{n_i}$, a function $g_\sigma : [m] \to [n]$ (which need not be a morphism in the simplicial category).

$$g_\sigma(i) = \begin{cases} \sigma_i(n_i) & \text{if } \sigma_i \neq d_k(id_n) \text{ or } id_n \\ k & \text{otherwise.} \end{cases}$$

We observe that g_σ induces a simplicial map $\operatorname{sd}\triangle^m \to \operatorname{sd}\triangle^n$. One can check that the image of this map is in $\operatorname{sd}\Lambda^n_k$.

2. Since acyclic cofibrations are closed under infinite composition, we need only prove that the map $j_X : X \to \operatorname{Ex}(X)$ is an acyclic cofibration. We omit the proof of this and invite the interested reader to look at [13].

3. We shall first prove that Ex preserves fibrations. By adjointness, this is true if and only if sd preserves acyclic cofibrations. One can prove (omitted here, see chapter 2 of [13]) that, in the category of simplicial sets, the acyclic cofibrations are generated by the horn inclusions $\varphi : \Lambda^n_k \to \triangle^n$. Thus it suffices to observe that the map

$$\operatorname{sd}\varphi : \operatorname{sd}\Lambda^n_k \to \operatorname{sd}\triangle^n$$

is an acyclic fibration. Observe first that it is obviously injective and thus a cofibration. Secondly, observe that, on the level of geometric realizations, we have the following diagram

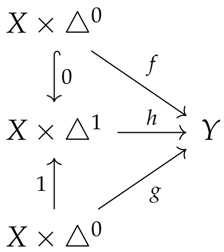
$$\begin{array}{ccc} |\operatorname{sd}\Lambda^n_k| & \xrightarrow{|\operatorname{sd}\varphi|} & |\operatorname{sd}\triangle^n| \\ \big\downarrow\cong & & \big\downarrow\cong \\ |\Lambda^n_k| & \xrightarrow{|\varphi|} & |\triangle^n| \end{array}$$

with all but the top horizontal map being known weak equivalences in the Quillen model structure. Therefore by the two out of three property, applied twice, we deduce that $|\operatorname{sd}\varphi|$ is a weak equivalence, and so $\operatorname{sd}\varphi$ is a weak equivalence in the Kan-Quillen model structure as desired. So Ex preserves fibrations f. Since Λ^n_k and $\triangle^n$ are small, $\operatorname{Ex}^\infty(f)$ has the right lifting property against morphisms of the form $\Lambda^n_k \to \triangle^n$ and so is a fibration.

4. Ex has a left adjoint and thus preserves all small limits. Finite limits commute with filtered colimits and so Ex preserves finite limits.
5. The morphism $\eta_X : X \to \mathrm{Ex}^\infty(X)$ is a weak homotopy equivalence. Therefore $\mathrm{Ex}^\infty(j_X) : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(\mathrm{Ex}^\infty(X))$ is a weak equivalence of fibrant objects and so is a homotopy equivalence. □

We shall now outline a slightly more complex, but equivalent, way to define the Kan–Quillen model structure on $\mathrm{Set}_\triangle$ that has the advantage of not making reference to topological spaces. We shall define the homotopy groups of a Kan complex. We extend our notion of homotopy groups to an arbitrary simplicial set X by saying it has the same homotopy groups as the fibrant object $\mathrm{Ex}^\infty(X)$.

Definition 3.4.12 Two simplicial morphisms $f, g : X \to Y$ are (left) homotopic, written $f \sim g$, if there exists a $H : X \times \triangle^1 \to Y$ making the following diagram commute



Remark 3.4.13 This definition can be applied directly, but it is sometimes more useful to reformulate it as follows. A homotopy between two maps is a family of functions $h_i : X_n \to Y_{n+1}$ for each integer $0 \leq i \leq n$ satisfying the following four conditions.

- $d_0 h_0 = f_n$.
- $d_{n+1} h_n = g_n$.
- $d_i h_j = \begin{cases} h_{j-1} d_i & \text{if } i < j. \\ d_i h_{i-1} & \text{if } i = j \neq 0. \\ d_i h_i & \text{if } i = j+1 \\ h_j d_{i-1} & \text{if } i > j+1. \end{cases}$
- $s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \leq j. \\ h_j s_{i-1} & \text{if } i > j. \end{cases}$

In the language of Definition 2.1.9, we are taking $X \times \triangle^1$ as a natural choice of cylinder.

Remark 3.4.14 Under the Quillen model category structure on $\mathrm{Set}_\triangle$, every object is cofibrant. Thus $\sim$ is an equivalence relation on $\mathrm{Hom}_{\mathrm{Set}_\triangle}(X, Y)$ if Y is fibrant.

Definition 3.4.15 Two simplicial sets X and Y are homotopy equivalent if there exist simplicial morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{Id}_Y$ and $g \circ f \sim \mathrm{Id}_X$.

To define the homotopy groups of a simplicial set, we will need a finer notion of homotopy equivalence.

Definition 3.4.16 Let X be simplicial set and let $W \subset X$. Then two simplicial morphisms $f, g : X \to Y$ are are *homotopic relative to* W if there exists a homotopy H such that $H|_{W \times \triangle^1}(w, t) = f(w) = g(w)$ for all t.

Definition 3.4.17 Let X be a fibrant simplicial set, $v \in X_0$ be a vertex and $n \geq 1$. Then the n^{th} *simplicial homotopy group* $\pi_n(X, v)$ is the set of simplicial functions $\triangle^n \to X$ which are constantly equal to v on $\partial \triangle^n$ modulo the homotopy equivalence relation.

Definition 3.4.17 describes the structure of $\pi_n(X, v)$ as a set. We still need to explain the precise group structure on X. To that end, let $\alpha, \beta : \triangle^n \to X$ represent elements of $\pi_n(X, v)$. Then, we define a $n+1$-tuple $\omega^{(\alpha,\beta)} = (v_0, v_1, \ldots v_{n-1}, \hat{v_n}, v_{n+1})$ (where $\hat{x}$ once again denotes the omission of x) of n-simplices of X as follows

$$\omega_i = \begin{cases} v & \text{for } 0 \leq i \leq n-2 \\ \alpha & \text{for } i = n-1 \\ \beta & \text{for } i = n+1 \end{cases}$$

Observe that $d_i \omega_j = d_{j-1} v_i$ if $i < j$ and $i, j \neq n$. Thus the tuple ω determines a simplicial morphism $\overline{\omega} : \Lambda^{n+1}_n \to X$ which sends the i^{th} face of Λ^{n+1}_n to ω_i. Since X is a fibrant set, $\overline{\omega}$ extends to a simplicial morphism $\theta : \triangle^{n+1} \to X$. Observe that

$$\begin{aligned} \partial(d_n\theta) = (d_0 d_n \theta, \ldots, d_{n-1} d_n \theta, d_n d_n \theta) &= (d_{n-1} d_0 \theta, \ldots, d_{n-1} d_{n-1} \theta, d_n d_n \theta) \\ &= (v, \ldots, v, v) \end{aligned}$$

so $d_n\theta$ represents an element of $\pi_n(X, v)$. One can easily check the following lemma.

Lemma 3.4.18 *[13, Lemma I.7.1] The homotopy class of $d_n\theta(\mathrm{rel}\, \partial \triangle^n)$ is independent of the choices of representatives of $[\alpha]$ and $[\beta]$, and the choice of lift θ.*

Therefore we have a well-defined binary operation

$$\begin{aligned} m : \pi_n(X, v) \times \pi_n(X, v) &\to \pi_n(X, v) \\ ([\alpha], [\beta]) &\mapsto [d_n \theta]. \end{aligned}$$

Moreover let $e \in \pi_n(X, v)$ be the equivalence class of the morphism

$$\triangle^n \to \triangle^0 \xrightarrow{v} X.$$

Theorem 3.4.19 *[13, Theorem I.7.2] Let X be a fibrant simplicial set and let v be 0-simplex of X. Then, with the definitions as above, $(\pi_n(X, v), m)$ is a group with identity e for $n \geq 1$. Moreover if $n \geq 2$, then $(\pi_n(X, v), m)$ is abelian.*

As promised earlier in the chapter, an important use of the Ex^∞-functor is that it gives a purely combinatorial description of the Kan-Quillen model structure on $\mathrm{Set}_\triangle$, without needing to refer to topological spaces.

Theorem 3.4.20 (Quillen) *The Kan-Quillen structure on the category $\mathrm{Set}_\triangle$ is equivalent to the structure given as follows*

- *The simplicial morphism $f : X \to Y$ is a weak equivalence if $\mathrm{Ex}^\infty(f) : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$ induces an isomorphism of homotopy groups.*
- *The fibrations are the Kan fibrations.*
- *The cofibrations are the levelwise injective simplicial morphisms.*

Remark 3.4.21 The statement of theorem 3.2.10 is almost exactly the same as the above theorem, except that we are implicitly using $S_\bullet(|X_\bullet|)$ as an alternative fibrant replacement functor. In the model structure on $\mathrm{Set}_\triangle$ this is always weakly homotopic to $\mathrm{Ex}^\infty(X_\bullet)$, but the same only if $X_\bullet$ is the disjoint union of points.

3.5 The Barratt-Eccles E_n-operad

The Barratt-Eccles E_n-operad provides us with a convenient model for the little n-discs operad. We shall produce it from a filtration of the *simplicial operad*, which itself is a simplicial model for the $\mathbb{D}_\infty$ operad. To understand the connection with the little discs operad it will then be necessary to introduce the idea of *cellular decomposition* which exhibits the desired equivalence on the level of geometric realization. The standard treatment of these things, which we are broadly following, is [2].

Definition 3.5.1 Let A be a set. We define *universal bundle functor* $W : \mathrm{Set} \to \mathrm{Set}_\triangle$ is on objects by $(W(A))_n := A^{\times(n+1)}$ equipped the with face and degeneracy maps

$$d_i(a_0, \dots, a_n) = (a_0, \dots, a_{i-1}, \hat{a}_i, a_{i+1}, \dots, a_n)$$
$$s_i(a_0, \dots, a_n) = (a_0, \dots, a_{i-1}, a_i, a_i, a_{i+1}, \dots, a_n)$$

where the notation $\hat{x}$ signifies the omission of x. On morphism $f : A \to B$ we define

$$W(f) : A \to B$$

$$(a_0, \dots, a_n) \mapsto (f(a_0), \dots, f(a_n))$$

Remark 3.5.2 The universal bundle functor is of independent interest in simplicial theory. In brief, suppose we have a topological group G which is *well pointed*; that is, the unique map to it from the trivial topological group is closed and a cofibration. Then the geometric realization of $W(G)$ has the same homotopy type as the classifying space of G.

Rather than working with operads, we will be using the more functorial notion of a preoperad.

Definition 3.5.3 Let Λ be the category with the nonempty finite ordinals $\mathbf{n} = \{1, \dots, n\}$ for objects and the injective maps between them for morphisms. For $i_1, \dots i_n \in \mathbf{m}$ we shall denote the morphism in $\mathrm{Hom}_\Lambda(\mathbf{n}, \mathbf{m})$ sending $(1, 2 \dots n)$ to $(i_1, i_2 \dots, i_n)$ by $\theta_{i_1 \cdots i_n}$. A *preoperad* taking values in a category C is a (contravariant) functor $\mathcal{O}_\bullet : \Lambda^{op} \to C$. We shall use the notation $\mathcal{O}_n$ for the object $\mathcal{O}(\mathbf{n}) \in C$ and φ^* for the morphism $\mathcal{O}(\varphi)$. A *morphism of preoperads* is a natural transformation of functors.

Remark 3.5.4 The category Λ has the same objects as the simplex category but not the same morphisms. Also, be warned that we are using a slightly different notation for ordinals in this definition than we did in Section 2.1.

When we were studying simplicial sets we remarked that every morphism in $\triangle$ could be decomposed uniquely into a increasing surjection followed by an increasing injection. Correspondingly every morphism $\theta : \mathbf{n} \to \mathbf{m}$ in Λ decomposes uniquely as a bijection $\theta^\sharp$ followed by an increasing map θ^{incr}.

We should think about preoperads as $\mathbb{S}$-modules with an additional 'almost simplicial' structure. This is best illustrated by the following example.

Example 3.5.5 Every *pointed (or unital) operad* $\mathcal{P}$, ie. operad with a distinguished 0–arity element $*$, has a canonical associated preoperad $\mathcal{O}$ that we construct as follows. On objects we set $\mathcal{O}_n := \mathcal{P}(n)$ for $n \geq 1$. Now let $\theta : \mathbf{n} \to \mathbf{m}$ be a morphism in Λ. Of course $\theta^\sharp$ is a bijective map $\mathbf{n} \to \mathbf{n}$ and thus can be viewed as an element $\sigma \in \mathbb{S}_n$. For $i \in \mathbf{m}$ we define

$$x_i = \begin{cases} 1 \in \mathscr{P}(1) & \text{if } i \in \mathrm{Im}(\theta) \\ * \in \mathscr{P}(0) & \text{if } i \notin \mathrm{Im}(\theta) \end{cases}$$

We can now define $\mathcal{O}$ on morphisms as

$$\theta^* : \mathcal{O}_m \to \mathcal{O}_n$$

$$x \mapsto \gamma(x, x_1, \dots, x_m) * \sigma^{-1}.$$

All the preoperads that we shall see in this chapter are operads. However, the notion of a preoperad is not strictly weaker than an operad. As we saw in the last example, one generally needs some extra structure eg. a distinguished arity 0 operation, to produce a preoperad from an operad.

Example 3.5.6 The *symmetric preoperad* $\mathbb{S} : \Lambda^{op} \to \mathrm{Set}$ is defined on objects by $\mathbb{S}_n := \mathrm{Hom}_\Lambda(\mathbf{n}, \mathbf{n})$. We identify this with the n^{th} symmetric group, thereby avoiding all possible confusion over notation. On morphisms $\theta \in \mathrm{Hom}_\Lambda(\mathbf{n}, \mathbf{m})$ we define

$$\theta^* : \mathrm{Hom}_\Lambda(\mathbf{m}, \mathbf{m}) \to \mathrm{Hom}_\Lambda(\mathbf{n}, \mathbf{n})$$

$$\sigma \mapsto (\sigma \circ \theta)^\sharp$$

The symmetric preoperad can be endowed with the operadic structure of the associative operad. This is given by the unit id $\in \mathbb{S}_1$ and by

$$\gamma : \mathbb{S}(r) \otimes \mathbb{S}(n_1) \otimes \cdots \otimes \mathbb{S}(n_r) \to \mathbb{S}(n_1 + \cdots + n_r) \tag{3.10}$$

$$(\sigma, \sigma_1, \ldots, \sigma_r) \mapsto \sigma_{n_1 \cdots n_r} \circ (\sigma_1 \times \cdots \times \sigma_r) \tag{3.11}$$

where $\sigma_{n_1,\ldots,n_r}$ is the permutation that operates on $\{1, 2, \ldots, n_1 + \cdots + n_r\}$ by breaking it into n blocks with the i^{th} of size n_i and permuting these blocks by σ.

Once we know what the symmetric preoperad is we can immediately write down the definition of the *simplicial preoperad* $\Gamma : \Lambda \to \mathrm{Set}_\triangle$ as the composition $W \circ \mathbb{S}$. One can check quite easily that the universal bundle functor respects Cartesian products. Therefore the operad structure on the symmetric preoperad extends to the simplicial preoperad.

Definition 3.5.7 The *simplicial operad* Γ is the preoperad $W \circ \mathbb{S}$ inheriting the operad structure on $\mathbb{S}$.

Remark 3.5.8 It follows easily from Remark 3.5.2 that $|\Gamma(n)|$ is the universal principal bundle for $\mathbb{S}_n$.

We shall briefly outline what this means combinatorially. The simplicial sets defining the simplicial operad in each arity are of the form

$$\Gamma(r)_n = \{(w_0, \ldots, w_n) \in \mathbb{S}_r \times \cdots \times \mathbb{S}_r\}$$

equipped the with face and degeneracy maps

$$d_i(w_0, \ldots, w_n) = (w_0, \ldots, w_{i-1}, \hat{w}_i, w_{i+1}, \ldots, w_n)$$
$$s_i(w_0, \ldots, w_n) = (w_0, \ldots, w_{i-1}, w_i, w_i, w_{i+1}, \ldots, w_n).$$

$\mathbb{S}_r$ acts on $\Gamma(r)$ diagonally, that is to say if $\sigma \in \mathbb{S}_r$ and $(w_0, \ldots, w_r) \in \Gamma(r)$ the

$$(w_0, \ldots, w_r) * \sigma = (w_0 * \sigma, \ldots, w_r * \sigma)$$

Finally the compositions are also defined componentwise via the explicit composition law of Equation 3.10 (in other words we view componentwise composition as occurring within the associative operad).

Remark 3.5.9 One can show that $\Gamma(n)$ is contractible for all n. Therefore the terminal morphism $\Gamma \to \mathrm{Comm}$ is a weak equivalence. In particular, the simplicial operad is an E_∞ operad.

Definition 3.5.10 Let k be a positive integer. The *Barratt-Eccles E_k-operad* is defined by

$$\Gamma^{(k)}(n) = \{x \in \Gamma(n) : \theta^*_{ij}(x) \in \mathrm{sk}_{k-1}\,\Gamma(2) \text{ for all } i < j\}.$$

where sk_{k-1} denotes $(k-1)$-skeleton and the notation θ_{ij} is defined in Definition 3.5.3.

One can show that $\Gamma^{(k)}$ defines a filtration of Γ, called the *Smith filtration*

$$\Gamma^{(1)} \hookrightarrow \Gamma^{(2)} \hookrightarrow \cdots \hookrightarrow \Gamma^{(i)} \cdots \hookrightarrow \Gamma^{(\infty)} = \Gamma$$

This chain of inclusions should be seen as the analogue in simplicial sets of Diagram 2.1.

Remark 3.5.11 Using the combinatorial description of the simplicial operad, we can give a more explicit description of the Smith filtration. Let τ be a permutation in $\mathbb{S}_r$. For $i < j$, let $\tau|_{i,j}$ be 0 if $\tau(i) < \tau(j)$ and 1 otherwise.

Consider a simplex $\sigma = (\sigma_0, \ldots, \sigma_n)$ in $\Gamma(r)_n$. Let μ^σ_{ij} be the number of times the sequence $(\sigma_0|_{i,j}, \ldots, \sigma_n|_{i,j})$ changes values. Then $\Gamma^{(k)}(r)_n$ is the $\sigma \in \Gamma(r)_n$ such that $\mu^\sigma_{ij} < k$ for all $i < j$.

The second half of this section is devoted to showing why the Barratt-Eccles operads act as a simplicial model for the little n-discs operad. This will mean defining the notion of an E_n-cellular operad. We shall see that the geometric realisation of the Barratt-Eccles E_n-operad has this structure. We shall then apply a theorem of Fiedorowicz states that any that cellular E_n-operads are E_n-operads, in the sense of Remark 2.4.2. We shall start this program by defining cells and cellular decomposition.

Definition 3.5.12 Let P be a poset and X be a topological space. A collection $(c_p)_{p \in P}$ of closed contractible subspaces, which we call *cells*, of X is called a *cellular P-decomposition* of X if it satisfies the following three conditions.

- $c_{p_1} \subseteq c_{p_2}$ if and only if $p_1 \leq p_2$.
- These inclusions are cofibrations in the Quillen model structure on Top.
- X is equal to the union of its cells and has the weak topology with respect to its cells, that is $X = \mathrm{colim}_P\, c_p$.

The *interior* of the cell c_p is

$$c_p^\circ = c_p \backslash (\bigcup_{q<p} c_q).$$

Of course, to obtain a E_n-cellular operad we will need to specify a particular poset. The PO preoperad is convenient way to parameterize this data.

Definition 3.5.13 Let $\mathbb{N}^{\binom{n}{2}}$ be the Cartesian product of $\binom{n}{2}$ copies of $\mathbb{N}$. We shall label the components of elements $\mu \in \mathbb{N}^{\binom{n}{2}}$ by ordered pairs of indices (i,j) with $1 \leq i < j \leq n$. Combinatorially we can view an element of $\mathbb{N}^{\binom{n}{2}}$ as the complete graph on n vertices with edges labelled by positive integers. We use this to define a preoperad structure, called the *complete graph preoperad*, as follows. On objects $\mathcal{O}_n := \mathbb{N}^{\binom{n}{2}}$. On morphisms $\theta \in \mathrm{Hom}_\Lambda(\mathbf{n},\mathbf{m})$ we define

$$\theta^* : \mathbb{N}^{\binom{m}{2}} \to \mathbb{N}^{\binom{n}{2}}$$

$$\theta^*(\mu)_{i,j} = \begin{cases} \mu_{\theta(i),\theta(j)} & \text{if } \theta(i) < \theta(j) \\ \mu_{\theta(j),\theta(i)} & \text{if } \theta(i) > \theta(j) \end{cases}$$

The complete graph preoperad has an operadic structure. Let $\mu \in \mathcal{O}_n$ and $\mu_i \in \mathcal{O}_{i_n}$ for $0 \geq i \geq n$. We can think of $\gamma(\mu,\mu_1,\ldots,\mu_n)$ as the complete graph on $i_1 + \cdots + i_n$ vertices. The edge labelling is determined as follows.

$$\gamma(\mu,\mu_1,\ldots,\mu_n)_{j,k} = \begin{cases} (\mu_r)_{\varphi_r^{-1}(j)\varphi_r^{-1}(k)} \text{ if } j,k \in \varphi_r(\mathbf{i_r}) \\ \mu_{r,s} \text{ if } j \in \varphi(\mathbf{i_r}) \text{ and } k \in \varphi_s(\mathbf{i_s}) \end{cases}$$

Here φ_r is the inclusion map $\mathbf{i_r} \hookrightarrow \mathbf{i_1} + \cdots + \mathbf{i_n}$. By $\mathbf{i_1} + \cdots + \mathbf{i_n}$ we mean the set $\{1,2,\ldots,i_1 + \cdots + i_n\}$ and the inclusion is thus given by sending $i \in \mathbf{i_r}$ to $i_1 + \cdots + i_{r-1} + i$.

Definition 3.5.14 We define the *PO preoperad* to be the Cartesian product of the symmetric preoperad with the complete graph preoperad. More precisely we define $\mathcal{K}_n = \mathbb{N}^{\binom{n}{2}} \times \mathbb{S}_n$. On morphisms $\theta \in \mathrm{Hom}_\Lambda(\mathbf{n},\mathbf{m})$ we define

$$\theta^* : \mathcal{K}_m \to \mathcal{K}_n$$

$$(\mu,\sigma) \mapsto (\theta^*_{cgp}(\mu), \theta^*_{ms}(\sigma))$$

where $\theta^*_{cgp} : \mathbb{N}^{\binom{m}{2}} \to \mathbb{N}^{\binom{n}{2}}$ and $\theta^*_{ms}(\sigma) : \mathbb{S}_n \to \mathbb{S}_n$ are the functions obtained by applying complete graph preoperad and the symmetric preoperad (which are functors) respectively to θ.

Remark 3.5.15 The PO in the name of the PO preoperad stands for partially ordered, and indeed each $\mathcal{K}_n$ is a poset. We say that $(\mu,\sigma) \leq (\nu,\tau)$ if for all $i < j$ either $\theta^*(\mu,\sigma) = \theta^*(\nu,\tau)$ or $\mu_{ij} < v_{ij}$.

An E_n-cellular preoperad is an operad which in the following precise sense admits a cellular decomposition partially ordered by the PO-preoperad.

Definition 3.5.16 A preoperad $\mathcal{O}$ over Top is called a *cellular E_∞-preoperad* if the $\mathbb{S}_2$–space $\mathcal{O}_2$ has a cellular $\mathcal{K}_2$–decomposition $(\mathcal{O}_2^{(p)})_{p \in \mathcal{K}_2}$ (in the sense of Definition 3.5.12 and Remark 3.5.15), where the actions of $\mathbb{S}_2$ on $\mathcal{O}_2$ and $\mathcal{K}_2$ are compatible. We require that this decomposition satisfies the two following conditions.

- For all $n > 0$ and all $p \in \mathcal{K}_n$, the object

$$\mathcal{O}_n^{(p)} := \bigcap_{1 \leq i < j \leq n} (\theta_{ij}^*)^{-1}(\mathcal{O}_2^{\theta_{ij}^*(p)}),$$

  is contractible, and for all $p, q \in \mathcal{K}_n$, with $p \leq q$ the natural inclusion $\mathcal{O}_n^{(p)} \subseteq \mathcal{O}_n^{(q)}$ is a cofibration in the Quillen model structure on Top. In other words $(\mathcal{O}_n^{(p)})_{p \in \mathcal{K}_n}$ forms a cellular $\mathcal{K}_n$-decomposition of $\mathcal{O}_n$ in the sense of Definition 3.5.12.

- Each $\mathbb{S}_n$–orbit of $\mathcal{O}_n$ contains an *ordered point*, that is a point $x \in \mathcal{O}_n$ whose projections $\theta_{ij}^*(x)$ are in the interior of cells of the form $\mathcal{O}_p^{(\mu, id_2)}$.

Definition 3.5.17 A reduced operad $\mathscr{P}$ is called a *cellular E_∞-operad* if the underlying preoperad is cellular E_∞, and such that the operad composition map is preserves the cellular structure, in the sense that

$$\gamma(\mathscr{P}_n^{(\mu,\sigma)} \times \mathscr{P}_{i_1}^{(\mu_1,\sigma_1)} \times \cdots \times \mathscr{P}_{i_n}^{(\mu_n,\sigma_n)}) \subseteq \mathscr{P}_{i_1+\cdots+i_n}^{(\gamma(\mu,\mu_1,\ldots,\mu_n),\gamma(\sigma,\sigma_1,\ldots,\sigma_n))}$$

The complete graph operad possess a natural filtration

$$\mathcal{K}_n^s = \{(\mu, \sigma) : \mu_{ij} < s \text{ for all } i < j\} \tag{3.12}$$

This induces the following filtration any given cellular E_∞–preoperad $\mathcal{O}$

$$\mathcal{O}^{(k)} = \bigcup_{q \in \mathcal{K}_n^k} \mathcal{O}_n^{(q)}.$$

Preoperads of the form $\mathcal{O}^{(k)}$ are called *cellular E_k–preoperads*, and operads which have underlying preoperads of the form $\mathcal{O}^{(k)}$ are called *cellular E_k–operads*.

Example 3.5.18 We can show that, up to weak equivalence of operads, the little n-discs operad is a cellular E_n-operad. We construct the cellular decomposition as follows.

First we replace the little n-discs operad with the little n-cubes operad $\mathcal{C}_n$(see Remark 2.4.8), to which it is weakly equivalent. In particular, we shall study the E_∞-little cubes operad, $\mathcal{C}_\infty$. We recall that this is defined as the colimit

$$\mathcal{C}_1 \hookrightarrow \mathcal{C}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_i \hookrightarrow \cdots \hookrightarrow \mathcal{C}_\infty.$$

where the embeddings are all along the equator.

For $(c_1, c_2) \in \mathcal{C}_\infty(2)$, we write $c_1 \sim_\mu c_2$ if there exists an $i \leq \mu + 1$ such that c_1 and c_2 are separated by a hyperplane perpendicular to the i^{th}-axis. If the first i such that is true is $\mu + 1$ itself, we further require that the cube c_1 lies on the negative side of $H_{\mu+1}$ and c_2 lies on the positive side. In this very particular case, we write $c_1 \sim'_\mu c_2$. It is now easy to define the cell structure on $\mathcal{C}_\infty(n)$. For $(\mu, \sigma) \in \mathcal{K}_n$ we let

$$\mathcal{C}_\infty(n)^{(\mu,\sigma)} = \{(c_1, c_2, \ldots, c_n) : c_i \sim_{\mu_{ij}} c_j \text{ if } \sigma(i) < \sigma(j) \text{ and } c_j \sim_{\mu_{ij}} c_i \text{ if } \sigma(j) < \sigma(i)\}.$$

Almost all the requirements for $\mathcal{C}_\infty$ to be an E_∞-cellular operad are now clearly satisfied. The only property that takes some work is verifying that these cells $\mathcal{C}_\infty(n)^{(\mu,\sigma)}$ are contractible.

To prove this it is necessary to distinguish between the case where the interior of the cell is empty (the cell is *proper*), and that when it is not (*improper*). We shall focus on the latter case first. Let $(c_1, \ldots, c_n)$ be an interior point of $\mathcal{C}_\infty(n)^{(\mu,\sigma)}$. Then, $c_i \sim'_{\mu_{ij}} c_j$ if $\sigma(i) < \sigma(j)$ and $c_j \sim'_{\mu_{ij}} c_i$ if $\sigma(j) < \sigma(i)$. Now, if $(\mu, \sigma) \in \mathcal{K}_n^{(p)}$, consider the natural projection

$$\pi : \mathcal{C}_\infty(n)^{(\mu,\sigma)} \to \mathcal{C}_\infty(n)^{(\mu,\sigma)} \cap \mathcal{C}_p(n).$$

This is a Serre fibration with contractible fibres, and therefore, by the long exact sequence in homotopy, we need only show that the image is contractible. To this end, note that we can contract the whole image to $\pi((c_1, \ldots, c_n)) = (\bar{c_1}, \ldots, \bar{c_p})$ in n coordinate-wise steps, where we start with the last coordinate and end with the first, and we deform in affine manner. Our choice of order here ensures that the appropriate hyperplanes separating cubes exist at all times, and so the contraction always stays within $\mathcal{C}_\infty(n)^{(\mu,\sigma)}$.

It remains to be shown that the cells with no interior are contractible. Indeed, something stronger is true; each cell $\mathcal{C}_\infty(n)^{(\mu,\sigma)}$ contains proper subcell $\mathcal{C}_\infty(n)^{(\hat{\mu},\hat{\sigma})}$, such that

$$\mathcal{C}_\infty(n)^{(\mu,\sigma)} = \mathcal{C}_\infty(n)^{(\hat{\mu},\hat{\sigma})}.$$

The proof of this is left as an exercise to the reader.

The projection π is thus a weak equivalence between the suboperad $\mathcal{C}_\infty^{(n)}$ and $\mathcal{C}_n$, as desired.

Proposition 3.5.19 *For each k, the geometric realization of the Barratt-Eccles E_k-operad is a cellular E_k-operad, In particular, the simplicial operad is realized as a cellular E_∞-operad.*

Proof We are going to construct a decomposition of the simplicial operad into simplicial subsets. When we pass to the geometric realisation, these shall form cells that exhibit $|\Gamma|$ as a cellular E_∞-operad.

To start, recall that each simplex $x \in \Gamma(r)_n$ is an $n+1$–tuple of elements of $\mathbb{S}_r$. We denote the last element in this tuple by σ_x. Then we have a decomposition of Γ, indexed by $(\mu, \sigma) \in \mathcal{K}_p$, given by

$$\Gamma(p)^{(\mu,\sigma)} = \{x \in \Gamma_p : \forall i < j, \theta^*_{ij}(x) \in \operatorname{sk}_{\mu_{ij}} \Gamma(2) \text{ and } \theta^*_{ij}(x) = \theta^*_{ij}(\sigma_x) \text{ if } \theta^*_{ij}(x) \notin \operatorname{sk}_{\mu_{ij}-1} \Gamma(2)\}.$$

These simplicial subsets are realizable as cells in the sense of Definition 3.5.12. This is because cell-inclusions are cofibrations in the Kan-Quillen model structure and hence their geometric realizations must also be cofibrations. Furthermore, we have a simplicial contraction in the sense of Remark 3.4.13 which is given by considering the function

$$h : \Gamma(p)^{(\mu,\sigma)}_n \to \Gamma(p)^{(\mu,\sigma)}_{n+1}$$

$$(\sigma_0, \sigma_1, \ldots, \sigma_n) \mapsto (\sigma_0, \sigma_1, \ldots, \sigma_n, \sigma)$$

and then setting $h_i = h$ for $0 \le i \le n$. So we see the geometric realization of $\Gamma(p)^{(\mu,\sigma)}$ is contractible and so $(|\Gamma(p)^{(\mu,\sigma)}_n|)_{(\mu,\sigma)\in\mathcal{K}_n}$ forms a cellular $\mathcal{K}_n$-decomposition of $|\Gamma(p)|$. So we have verified the first condition of Definition 3.5.16.

To check the second condition, it suffices to observe that a point in $|\Gamma(p)|$ is ordered if it is contained in the realization of a simplex of $\Gamma(p)$ which has id_p as its last component. In particular, every $\mathbb{S}_p$-orbit of $|\Gamma(p)|$ contains one such point. Therefore $|\Gamma(p)|$ is a cellular E_∞ preoperad.

We now need to check that multiplication preserves cellular structure. It follows from the definition of the simplicial operad that $\gamma^\Gamma = W(\gamma^{\mathrm{Assoc}})$ (where W is the universal bundle construction). From this we easily obtain the following identities

$$\theta^*_{\varphi_r^{-1}(i),\varphi_r^{-1}(j)}(\gamma^\Gamma(x, x_1, \ldots, x_n)) = \theta^*_{i,j}(x_r) \text{ for } i, j \in \mathbf{i_r}.$$

$$\theta^*_{\varphi_r^{-1}(i),\varphi_s^{-1}(j)}(\gamma^\Gamma(x, x_1, \ldots, x_n)) = \theta^*_{r,s}(x) \text{ for } i \in \mathbf{i_r}, i \in \mathbf{i_s}, r < s.$$

Therefore cellular structure is preserved as desired and so the simplicial operad is a cellular E_∞-operad.

Lastly, the filtration on the simplicial operad induced by that of the graph operad is exactly the Smith filtration. So the Barratt-Eccles E_k-operad is a cellular E_k-operad as desired. □

The following theorem tells us that all cellular E_k-operads are essentially the same.

Theorem 3.5.20 (Fiedorowicz [2]) *Any two cellular E_k-operads are weakly equivalent as operads.*

Remark 3.5.21 We shall not write out a detailed proof of this theorem. The key idea though is that all cellular E_∞-operads admit a retract onto the geometric realization of the nerve of the complete graph operad $|\mathcal{N}(\mathcal{K})|$ (note that the nerve functor preserves limits, so this is indeed an operad), and thus we can always find a zig-zag between them.

As $|\Gamma^{(k)}|$ is a cellular E_k-operad, and $\mathbb{D}_k$ is weakly equivalent to one, it immediately follows from Theorem 3.5.20 that that little k-discs operad and the $|\Gamma^{(k)}|$ are weakly equivalent. Therefore, as promised earlier, the Barratt-Eccles E_k-operad is a small simplicial model of the little k-discs operad.

While we have been using the idea of weak equivalence for operads for quite a while now, we have neglected to fully explain the model structure behind it, or indeed why it is a useful notion. This gap in our understanding will be remedied in the next chapter.

Chapter 4

The homotopy theory of operads

Homotopy algebras were originally defined by Jim Stasheff [30] during the 1960s in order to study H-spaces. Since then, they have grown into a central object of interest, not just algebraic topology, but in modern mathematics as whole [17]. The principal goal of this section is to develop the theory of homotopy algebras over an operad. The best known examples of these are the A_∞-algebras, which are homotopy algebras over the associative operad. Intuitively these are algebras that are 'almost' associative, or, more precisely, are associative up to a series of coherent homotopies. This is a phenomenon that occurs quite a lot in nature. For example, path composition in classical homotopy theory is an operation that is not strictly associative but is only associative up to homotopy.

In order to define homotopy algebras over operads, we will need to first define a model structure on (well-behaved) operads. For any sufficiently well-behaved operad $\mathscr{P}$, we will be able to define a homotopy $\mathscr{P}$-algebra as an algebra over a cofibrant replacement of $\mathscr{P}$ within this model structure.

In order for a homotopy $\mathscr{P}$-algebra to be a tractable notion, it is thus necessary to be able to compute cofibrant replacements in this model category of operads. Fortunately, a convenient cofibrant replacement functor called the Boardman–Vogt resolution (or W-construction) exists, and is described in Section 4.3. In the modern theory of dendroidal sets [24], the Boardman–Vogt resolution is also used to extend the notion of a nerve from categories to operads. Unfortunately the details of this lie beyond the scope of this text.

The structure of our discussion is as follows. In Section 4.1, we shall describe how to transfer the model structure from an ambient category to the category of operads over it. We shall also define homotopy algebras. We shall follow this up in Section 4.2 with a brief detour into the land of free operads. These shall be necessary to construct the Boardman-Vogt resolution in Section 4.3. The final section (Section 4.4) will consist of explicit calculations, describing the Boardman-Vogt resolution in a number of cases; the associative operad in topological spaces (Subsection 4.4.1), the associative operad in simplicial sets (Subsection 4.4.2), and the Barrett-Eccles E_n-operad (Subsection 4.4.3).

4.1 The model category of operads

Operad theory always takes place over an ambient symmetric monoidal category. In this section we are going to show that any sufficiently nice symmetric monoidal category $(C, \otimes)$ equipped with compatible model category structure induces a canonical model structure on operads over it, called the *Berger-Moerdijk model structure*.

The first question to ask ourselves is; what conditions characterise a sufficiently nice symmetric monoidal category? 'Niceness' is encapsulated by the condition that the hom functor of any two objects should remain in the category.

Definition 4.1.1 Suppose that $(C, \otimes)$ is a symmetric monoidal category. An *internal hom* is a functor

$$[-,-] : C^{op} \times C \to C$$

such that, for all $X \in C$ the functor $[X, -]$ is left adjoint to the functor $- \otimes X$. In this case we say that C is *closed*.

We have already seen several examples of closed categories. Both Top and $\text{Set}_\triangle$ can be equipped with this structure by taking the internal hom to be their respective mapping spaces.

Let $(\mathbf{C}, \otimes, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a closed symmetric monoidal category equipped with a model structure. Compatibility between the model structure and the tensor product is captured by the *pushout-product condition*. This states that if $f : A \hookrightarrow B$ and $g : X \hookrightarrow Y$ are cofibrations then

$$A \otimes Y \coprod_{A \otimes X} B \otimes X \to B \otimes Y$$

is as well. Moreover this is an acyclic cofibration if and only if one of f or g is.

Definition 4.1.2 Let $(\mathbf{C}, \otimes, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a symmetric monoidal category equipped with a model structure. If $\mathbf{C}$ satisfies the pushout-product condition we say that $\mathbf{C}$ is a *symmetric monoidal model category*.

The pushout-product condition implies that a cofibration (or an acyclic cofibration) that is tensored with a cofibrant object remains a cofibration (or an acyclic cofibration).

To proceed we must to be able move a model category structure through an adjunction. Therefore we shall next introduce the *Transfer Principle* which is a list of sufficient conditions for this to be possible. One of these conditions is that the category is *cofibrantly generated*, a term which we now define.

Definition 4.1.3 An object A of a category C is said to be *small* if for all diagrams X in C of the form

$$X_1 \to X_2 \to \cdots \to X_i \to \cdots,$$

the natural functions $X_i \to \operatorname{colim} X$ induce a bijection

$$\operatorname{colim}_{n\in\mathbb{N}} \operatorname{Hom}_C(A, X_n) \to \operatorname{Hom}_C(A, \operatorname{colim}_{n\in\mathbb{N}} X_n).$$

Definition 4.1.4 A model category is *cofibrantly generated* if the category is cocomplete and admits a set of cofibrations and a set of acyclic fibrations, both with small domains such that the fibrations are characterized by their right lifting property with respect to the generating acyclic cofibrations and the acyclic fibrations are characterized by their right lifting property with respect to the generating cofibrations.

In general, almost all model structures of interest to mathematicians are cofibrantly generated. For example, both $\mathrm{Set}_\triangle$ and Top are.

Example 4.1.5 It is very easy to see that the category $\mathrm{Set}_\triangle$ with the Kan-Quillen model structure is cofibrantly generated. The generating cofibrations are the horn inclusions $\partial\triangle^n \hookrightarrow \triangle^n$ for $n \in \mathbb{N}$. The generating acyclic cofibrations are the horn inclusions $\Lambda^n_k \hookrightarrow \triangle^n$ for $n \in \mathbb{N}$ and $n \geq k \geq 0$. The geometric realisations of these morphisms also exhibit Top with the Quillen model structure as cofibrantly generated.

Definition 4.1.6 Let X be an object in a model category $\mathbf{C}$. A *path-object* for X is a factorisation of the diagonal

$$X \xrightarrow{\sim} \mathrm{Path}(X) \twoheadrightarrow X \times X$$

into a weak equivalence followed by a fibration.

We are now in a position to state the transfer principle.

Theorem 4.1.7 (Transfer Principle) *[4, Section 2.5] Let $F : \mathbf{C} \leftrightarrows D : G$ be an adjunction with $\mathbf{C}$ a cofibrantly generated model category and D a category with small colimits and finite limits. Define a map f in D to be a weak equivalence (resp. fibration) if and only if $G(f)$ is a weak equivalence (resp. fibration). Then provided that the following conditions are met this defines a cofibrantly generated model structure, called the transferred model structure, on D.*

- *The functor F preserves small objects.*
- *With respect to the definition of a fibration above in D above, D has functorial fibrant replacement.*
- *With respect to the definition of a fibration in D above, D has functorial path-object replacement for fibrant objects.*

Remark 4.1.8 The form of the transfer principle stated above can be improved considerably. In the first condition, small can be weakened to *λ-small* for any regular cardinal λ. The final two conditions can be replaced with the single requirement that any sequential colimit of pushouts of images under F of the generating acyclic cofibrations of $\mathbf{C}$ yields a weak equivalence in D. In practice these conditions are very difficult to verify directly, and that is why we use the formulation above.

Returning now to our main purpose, we shall use the Transfer Principle to construct a model structure on the category of $\mathbb{S}$-modules.

Let $(\mathbf{C}, \otimes, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a closed symmetric cofibrantly generated monoidal model category and G be a discrete group. We denote the category of objects of $\mathbf{C}$ equipped with a right G-action by $\mathbf{C}^G$. The morphisms in this category are the equivariant maps. The monoidal structure on $\mathbf{C}$ descends to $\mathbf{C}^{\mathsf{G}}$ via the *diagonal action* of G on the tensor product. It remains closed because G acts by conjugation on the internal hom.

Let $F : \mathbf{C}^{\mathsf{G}} \to \mathbf{C}$ be the forgetful functor. This admits a left adjoint, the free functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}^{\mathsf{G}}$. The model structure transfers across this adjunction as the conditions of the transfer principle are trivially satisfied. Thus $\mathbf{C}^{\mathsf{G}}$ has an induced model structure.

Taking $G = \mathbb{S}_n$ we see that from the above discussion we have a model structure on the category of $\mathbb{S}$-modules, viewed as the product category $\prod_{n \geq 0} \mathbf{C}^{\mathbb{S}_n}$. We transfer this structure to the category of operads using the forgetful functor – free functor adjunction between the categories of operads and $\mathbb{S}$-modules. Concretely, a morphism of operads $f : \mathscr{P} \to \mathscr{Q}$ is a fibration (resp. weak equivalence) if and only if the induced map $\mathscr{P}(n) \to \mathscr{Q}(n)$ is a fibration (resp. weak equivalence) in the category $\mathbf{C}^{\mathbb{S}_n}$ for all $n \in \mathbb{N}_0$.

Unfortunately, in general the theory we have developed behaves badly. In particular, given an operad $\mathscr{P}$ it is easy to see that the initial object in the category of $\mathscr{P}$-algebras is $\mathscr{P}(0)$. Therefore the category of $\mathscr{P}$-algebras under a given $\mathscr{P}$-algebra A is equivalent to the category of algebras over another operad $\mathscr{Q}$ such that $\mathscr{Q}(0) = A$. Thus the general homotopy theory of operads subsumes the homotopy theory of algebras over a given operad, which is well-known to be extremely badly behaved. To remedy this we will consider only *reduced operads*.

Definition 4.1.9 An operad $\mathscr{P}$ in a monoidal category C with unit I is said to be reduced if $\mathscr{P}(0) = I$. A morphism of reduced operads is an operad morphism $\Phi : \mathscr{P} \to \mathscr{Q}$ between reduced operads such that $\varphi(0)$ is the identity on I.

One might naively think that in a manner exactly analogous to that outlined above, we could transfer the homotopy structure from the ambient category C to that of reduced operads. For most part one would be correct. The only condition that presents difficulty, and in fact is not necessarily true, is the existence of functorial path-objects in the operad category. To ensure this, we require that C admits a *Hopf interval*.

Definition 4.1.10 A *Hopf object* in a symmetric monoidal category $(C, \otimes)$ is an object $(H, \mu, \nu, \triangle, \varepsilon)$ such that (H, μ, ν) is a monoid, $(H, \triangle, \varepsilon)$ a comonoid and μ, ν are maps of comonoids and $\triangle, \varepsilon$ are maps of monoids.

Example 4.1.11 The unit I in a symmetric monoidal category C is a Hopf object, by the canonical isomorphism $I \otimes I \to I$ and its inverse. The coproduct $I \sqcup I$ also possesses a Hopf object structure. Both the product and coproduct on $I \sqcup I$ are induced componentwise by the product and coproduct on I. One can check that the folding map $I \sqcup I \to I$ commutes with the relevant products and coproducts and is thus a morphism of Hopf objects. We say that C admits a *Hopf interval* if the folding map can be factored into a cofibration followed by a weak equivalence

$$I \sqcup I \hookrightarrow H \xrightarrow{\sim} I$$

where H is a Hopf object and both maps are morphisms of Hopf objects.

As the reader has probably already guessed from its definition, the existence of a Hopf interval allows one to construct a functorial path-object for fibrant objects. Everything we have stated and proved in this chapter can therefore by encapsulated in the following theorem.

Theorem 4.1.12 *[4, Theorem 3.1] Let $(\mathbf{C}, \otimes, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a closed symmetric monoidal cofibrantly generated model category with unit I such that*

1. *I is cofibrant*
2. *the over-category $\mathbf{C}/I$ has a symmetric monoidal fibrant replacement functor*
3. *$\mathbf{C}$ admits a commutative Hopf interval.*

Then there is a cofibrantly generated model structure on the category of reduced operads, in which a morphism of reduced operads $f : \mathscr{P} \to \mathscr{Q}$ is a fibration (resp. weak equivalence) if and only if the induced map $\mathscr{P}(n) \to \mathscr{Q}(n)$ is a fibration (resp. weak equivalence) in the category $\mathbf{C}^{S_n}$ for all $n \in \mathbb{N}_0$.

Remark 4.1.13 This theorem comes with a warning. To verify that a map is a fibration or a weak equivalence, or that an object is fibrant is very easy. We just need to apply the forgetful functor and then check the relevant property in the underlying category of $\mathbb{S}$-modules. **For questions of cofibrancy this does not work.** It is thus important to distinguish between two related notions of operadic cofibrancy. An operad can be cofibrant in the operadic model sense of Theorem 4.1.12, in which case we just call it *cofibrant*. However an operad $\mathscr{P}$ can have an cofibrant underlying $\mathbb{S}$-module but not necessarily be operadically cofibrant. We say that an operad that is cofibrant in this second sense is $\mathbb{S}$-*cofibrant*.

Remark 4.1.14 The proof of Theorem 4.1.12 is quite easy; one only needs to check the conditions of the Transfer Principle are satisfied. The fibrant replacement condition is immediate, one only needs to apply the fibrant replacement functor in each arity. Functorial path-object replacement requires convolution operads, which is a topic beyond the scope of this text. For a complete treatment, please see [4].

Theorem 4.1.12 is generally the situation of most interest in applications, because most operads seen in the everyday life of a mathematician are reduced. It is not the only situation in which the transfer principle may be employed though.

Definition 4.1.15 A monoidal category $(C, \otimes)$ is called *Cartesian* if the monoidal product coincides with the category-theoretical product.

Theorem 4.1.16 *[4, Theorem 3.2] Let $(\mathbf{C}, \otimes, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a symmetric, Cartesian, cofibrantly generated, closed model category with unit I such that*

1. *I is cofibrant*
2. *the over-category $\mathbf{C}/I$ has a symmetric monoidal fibrant replacement functor*

Then there is a cofibrantly generated model structure on the category of reduced operads, in which a morphism of reduced operads $f : \mathscr{P} \to \mathscr{Q}$ is a fibration (resp. weak equivalence) if and only if the induced map $\mathscr{P}(n) \to \mathscr{Q}(n)$ is a fibration (resp. weak equivalence) in the category $\mathbf{C}^{\mathbb{S}_n}$ for all $n \in \mathbb{N}_0$.

We end this section with the definition of an ∞-algebra (or a homotopy algebra). As mentioned in the chapter introduction, given an operad $\mathscr{P}$, we would like to study algebras that have an action of the $\mathscr{P}$ operad, but potentially not a strict one. The easiest example of this is one dimensional loop composition $\circ$. Given a topological space X and three based loops $\alpha, \beta, \gamma : S^1 \to X$, the 3-fold composites $(\alpha \circ \beta) \circ \gamma : [0,1] \to X$ and $\alpha \circ (\beta \circ \gamma) : [0,1] \to X$ will not be equal, but will be homotopic. When we start with 4 loops instead of 3, things get more interesting. Up to permutation, there are 5 ways we can compose them. We will have homotopies between every pair of composites, and *homotopies between these homotopies*. We can clearly continue like this. Informally, all this data, compositions, homotopies between compositions, homotopies between homotopies between compositions etc. determine a homotopy associative algebra. We shall return to this precise example in Subsection 4.4.1, For now, we shall see how to formalise this intuition.

Definition 4.1.17 Let $\mathscr{P}$ be a reduced operad over a category $\mathcal{C}$ satisfying the hypotheses of Theorem 4.1.12 (or alternatively Theorem 4.1.12). Let $\mathscr{P}^{cofib}$ be a cofibrant replacement for $\mathscr{P}$ in the model structure on operads. A *$\mathscr{P}_\infty$-algebra*, or a *homotopy $\mathscr{P}$-algebra*, is an algebra over $\mathscr{P}^{cofib}$.

Remark 4.1.18 This definition is motivated by Theorem 2.1.13. It is the usual trick of passing to the cofibrant replacement to ensure that every algebra morphism in the homotopy category is represented in the model category. We also usually require the endomorphism operad to be by definition fibrant.

4.2 The free operad

This section is devoted to giving the very concrete combinatorial description of the free functor between $\mathbb{S}$-modules and operads that we used above. This is of interest in its own right, but it also allows us to introduce a lot of the constructions, theory and examples that we will need in the next section when we study cofibrant replacements in the operad category.

The basic combinatorial objects we shall be working with are *planar trees*.

Definition 4.2.1 A *planar tree T* is a directed connected graph with no loops, which has the following properties.

- T has an output vertex and edge, that is a distinguished root vertex of valency 1 and its attached edge which must be incoming.
- T has some number of input vertices and edges, that is vertices of valency 1 and their attached edges which must be outgoing. These and the input vertices and edges are collectively referred to as *external*.
- Every other vertex and edge of T is referred to as *internal*. It must have exactly 1 outgoing edge.

Remark 4.2.2 Perhaps confusingly, it is standard in the literature to refer to internal vertices with exactly one **incoming** edge as *unary*. Such vertices of course also have an outgoing edge.

We illustrate the definition of a planar tree by drawing the third corolla t_3 in Figure 4.1. The only internal vertex is labelled with an i and the external

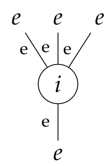


Figure 4.1: The corolla t_3

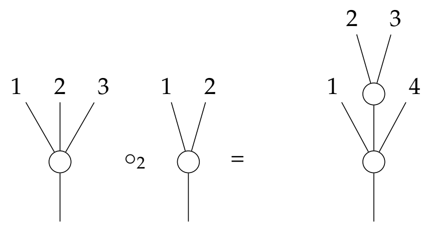


Figure 4.2: Partial composition in $\mathbb{T}$

edges are labeled with an e. Because external vertices are always attached to external edges, it is simpler to omit them. In the figure we have adopted this convention, but indicated where each missing vertex 'should' be with a floating e. We refer to the vertex that the output edge is outgoing from as the *internal root vertex*.

Planar trees give rise to one of the most important combinatorial operads, the *operad of planar trees*.

Definition 4.2.3 Let $\mathbb{T}$ be the operad in sets whose underlying $\mathbb{S}$-module structure in arity n is given by the collection of planar trees with exactly n input edges labelled from the set $\{1, \cdots, n\}$. The right symmetric action is given by permutation of these labels. Suppose that $T \in (r)$ and $T_i \in (n_i)$ for $r \geq i \geq 1$. Then the composition $\gamma(T, T_1, \ldots, T_n)$ is computed as follows.

1. For each i, let S^i be the tree isomorphic to T_i, but with its input edges labelled from the set $\{n_1 + \cdots, n_{i-1} + 1, \ldots n_1 + \cdots + n_i\}$. This is achieved by adding $n_1 + \cdots + n_{i-1}$ to the label of each input edge of T_i.

2. For each i, grafting the tree S^i onto T at the label i and replacing the joining $\rightarrow \bullet \rightarrow$ of two external edges at an external vertex with a single edge $\rightarrow$.

3. This produces the desired tree $\gamma(T, T_1, \ldots, T_n)$ which has input edges labelled from the set $\{1, n_1 + \cdots + n_r\}$.

The identity $I_{\mathbb{T}}$ is the tree in $\mathbb{T}(1)$ with no internal vertices, and with one output edge which is also the input edge.

One can look to figure 4.2 to see a visual example of partial composition of two elements of $\mathbb{T}$. It is clear that every tree in $\mathbb{T}$ can be built recursively by iterative composition starting with the identity tree $I_{\mathbb{T}}$ and the n-corollas t_n for all $n \geq 1$. This enables us to make inductive arguments about trees, something that will be a frequent trope in what will follow. The definition of the automorphism group illustrates this principle.

Definition 4.2.4 Let $T \in \mathbb{T}$ be a planar tree. Up to isomorphism, for some $k \in \mathbb{N}$, T can be represented as

$$T := \gamma(t_k, T_1^1, \dots, T_{i_1}^1, T_1^2, \dots, T_{i_2}^2, \dots, T_{i_n}^n)$$

where $T_1^k, \dots, T_{i_k}^k$ are copies of the same tree. Then the *automorphism group* $\mathrm{Aut}(T)$ is defined by first setting $Aut(t_n) = \mathbb{S}_n$ for all $n \in \mathbb{N}$, and then recursively defining

$$\mathrm{Aut}(T) := (\mathrm{Aut}(T_1^1)^{\times i_1} \times \cdots \times \mathrm{Aut}(T_1^n)^{\times i_n}) \rtimes (\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_n})$$

where $\mathbb{S}_{i_p}$ acts on $\mathrm{Aut}(T_1^p)^{\times i_p}$, for each p, by permuting the factors in the product.

Remark 4.2.5 It is important to gain an intuition for what the

$$T = \gamma(t_k, T_1^1, \dots, T_{i_1}^1, T_1^2, \dots, T_{i_2}^2, \dots, T_{i_n}^n)$$

decomposition means, as we will be using it a lot. Essentially, the internal root vertex and its immediate children form a copy of t_k within T. Meanwhile, each child vertex of the internal root vertex of T is the internal root vertex of a smaller subtree, which is one of the respective T_m^l.

One way to view the free m-ary operation on the data of an $\mathbb{S}$-module $\mathscr{P}$ is a tree T with m input edges and with each internal vertex decorated with elements of $\mathscr{P}(n)$, where n its the number of incoming edges. This should be viewed as using n-ary operation to compose the operations decorating those vertices from which these edges originate. The best way to formally codify this idea is the following definition.

Definition 4.2.6 Given any $\mathbb{S}$-module $\mathscr{P}$ over a category $\mathbf{C}$, one defines a (contravariant) functor $\overline{\mathscr{P}} : \mathbb{T}^{op} \to \mathbf{C}$ inductively as follows. We fix $\overline{\mathscr{P}}(I_{\mathbb{T}}) = I_{\mathbf{C}}$ and

$$\overline{\mathscr{P}}(\gamma(t_n, T_1, T_2, \dots, T_n)) = \mathscr{P}(n) \otimes \overline{\mathscr{P}}(T_1) \otimes \cdots \otimes \overline{\mathscr{P}}(T_n).$$

where t_n is the n-corolla. Given a tree isomorphism $f : T \to T'$ we decompose both trees as $T = \gamma(t_n, T_1 \dots, T_n)$ and $T' = \gamma(t_n, T_1', \dots, T_n')$ for some $n \geq 1$. The action of f decomposes into an action on t_n, denoted $\sigma \in \mathbb{S}_n$, which permutes the T_i, and a collection of tree isomorphisms $f_i : T_i \to T'_{\sigma(i)}$. We define

$$\overline{\mathscr{P}}(f) : \overline{\mathscr{P}}(n) \otimes \overline{\mathscr{P}}(T_1) \otimes \cdots \otimes \overline{\mathscr{P}}(T_n) \to \overline{\mathscr{P}}(n) \otimes \overline{\mathscr{P}}(T_{\sigma(1)}) \otimes \cdots \otimes \overline{\mathscr{P}}(T_{\sigma(n)})$$

as the morphism that acts by the right action of σ on $\mathscr{P}(n)$ and by the (recursively defined) morphism $\overline{\mathscr{P}}(f_i)$ on $\overline{\mathscr{P}}(T_i)$.

Theorem 4.2.7 *[5] The free operad $\mathcal{F}(\mathscr{P})$ can be computed as*

$$\mathcal{F}(\mathscr{P})(n) = \coprod_{[T], T \in \mathbb{T}(n)} \overline{\mathscr{P}}(T) \otimes_{Aut(T)} I[\mathbb{S}_n]$$

Here, $[T]$ ranges over the isomorphism classes of planar trees in $\mathbb{T}$, and $I[\mathbb{S}_n]$ is equal to $\coprod_{\sigma \in S_n} I_{\mathbf{C}}$ equipped with the obvious $\mathbb{S}$-action. The group $Aut(T)$ acts on $\mathbb{S}_n$ because a permutation of T induces a permutation of the labelled input vertices of T. This obviously extends to the action on $I[\mathbb{S}_n]$ that we see in the equation.

The operadic composition map

$$\gamma : \mathcal{F}(\mathscr{P})(r) \otimes \bigotimes_{i=1}^{r} \mathcal{F}(\mathscr{P})(n_i) \to \mathcal{F}(\mathscr{P})(n_1 + \cdots n_r)$$

is inclusion.

The following lemma follows from the pushout-product condition.

Lemma 4.2.8 *Let $\mathscr{P}$ be an $\mathbb{S}$-module. If $\mathscr{P}$ is cofibrant then $\mathcal{F}(\mathscr{P})$ is $\mathbb{S}$-cofibrant.*

We shall now introduce a slight variant of the free operad, which we shall use in the construction of Boardman-Vogt resolution in the next section. The extra condition that we shall need to impose is that the operad is *well-pointed*.

Definition 4.2.9 An $\mathbb{S}$-module $\mathscr{P}$ is said to be *pointed* if it is equipped with a base point $I \to \mathscr{P}(1)$. We write $\mathbb{S}_*\mathsf{Mod}$ for the category consisting of $\mathbb{S}$-modules and pointed morphisms between them. Pointed $\mathbb{S}$-modules such that the base point morphism $I \to \mathscr{P}(1)$ is a cofibration are called *well-pointed*. We also refer to operads as well-pointed if their underlying $\mathbb{S}$-module is.

As every operad has a unit there is a forgetful functor from operads to $\mathbb{S}_*\mathsf{Mod}$. We define the free pointed functor $\mathcal{F}_*$ as its left adjoint. The key difference between $\mathcal{F}_*$ and $\mathcal{F}$ is that the unit of $\mathcal{F}$ is added freely whereas for $\mathcal{F}_*$ the unit of the $\mathbb{S}$-module becomes the operadic unit. Thus, if we let $\underline{I}$ be the initial operad in $\mathbf{C}$ and $\hat{I}$ be the $\mathbb{S}$-module defined by $\hat{I}(i)$ equal to I if $i = 1$ and 0 otherwise (note that this is the underlying $\mathbb{S}$-module of $\underline{I}$), $\mathcal{F}_*$

will be the following pushout.

$$\begin{array}{ccc} \mathcal{F}(\hat{I}) & \longrightarrow & \mathcal{F}(\mathscr{P}) \\ \downarrow & & \downarrow \\ \underline{I} & \longrightarrow & \mathcal{F}_*(\mathscr{P}). \end{array} \tag{4.1}$$

One can easily prove the following.

Lemma 4.2.10 *For any well-pointed cofibrant* $\mathbb{S}$*-module* $\mathscr{P}$ *of a cofibrantly generated monoidal model category* $\mathbf{C}$*, the operad* $\mathcal{F}_*(\mathscr{P})$ *is cofibrant.*

Proof Since $\hat{I} \to \mathscr{P}$ is a cofibration of $\mathbb{S}$-modules, we know that $\mathcal{F}_*(\hat{I}) \to \mathcal{F}_*(\mathscr{P})$ is a cofibration of operads. The property of being a cofibration is invariant under pushouts, and so it follows from Diagram 4.1 that $\mathcal{F}_*(\mathscr{P})$ is cofibrant as an operad. □

4.3 The Boardman–Vogt resolution

As we saw in Section 4.1, a morphism of operads, over some ambient category $\mathcal{C}$, is fibrant or a weak equivalence if and only if the underlying morphism of $\mathbb{S}$-modules is fibrant or a weak equivalence. Therefore questions involving these can be dealt with using only the tools of $\mathcal{C}$. Cofibrancy is a less directly tractable property and thus we need an alternative way to understand it. This is also quite important from the perspective of explicitly constructing ∞-algebras. It turns out that for a special class of operads one can construct a cofibrant replacement functor called the *Boardman–Vogt resolution* or the *W-construction*.

As we shall see, the actual construction is extremely technical but the intuition behind it is fairly simple. We have already seen that, given an arbitrary pointed operad $\mathscr{P}$ that is cofibrant as an $\mathbb{S}$-module, $\mathcal{F}_*(F_*(\mathscr{P}))$ will be cofibrant as an operad, where F_* is the forgetful functor that is right adjoint to $\mathcal{F}_*$.

We are going to combinatorially construct a factorisation of the counit of the free–forgetful adjunction. With some assumptions on the operad $\mathscr{P}$, the factorisation

$$\mathcal{F}_*(F_*(\mathscr{P})) \hookrightarrow W(\mathscr{P}) \xrightarrow{\sim} \mathscr{P}$$

will be a weak equivalence followed by a cofibration. Thus $W(\mathscr{P})$ is a cofibrant replacement of $\mathscr{P}$.

Definition 4.3.1 An *interval* in $\mathbf{C}$ is a factorisation $I \sqcup I \hookrightarrow H \xrightarrow{\sim} I$ of the folding map into a cofibration $(0,1)$ followed by a weak equivalence

ε, together with an associative operation $\vee : H \otimes H \to H$ such that all of the following diagrams commute.

$$\begin{array}{ccccc} I \otimes H & \xrightarrow{0 \otimes H} & H \otimes H & \xrightarrow{H \otimes 0} & H \otimes I \\ & \searrow^{\sim} & \downarrow{\scriptstyle \vee} & \swarrow_{\sim} & \\ & & H & & \end{array}$$

$$\begin{array}{ccccccccc} & & I \otimes H & \xrightarrow{1 \otimes H} & H \otimes H & \xleftarrow{H \otimes 1} & H \otimes I & & \\ & \swarrow^{I \otimes \varepsilon} & \downarrow & & \downarrow{\scriptstyle \vee} & & \downarrow & \searrow^{\varepsilon \otimes I} & \\ I \otimes I & \xrightarrow{\sim} & I & \xrightarrow{1} & H & \xleftarrow{1} & I & \xleftarrow{\sim} & I \otimes I \end{array}$$

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\varepsilon \otimes \varepsilon} & I \otimes I \\ \downarrow{\scriptstyle \vee} & & \downarrow{\scriptstyle \sim} \\ H & \xrightarrow{\varepsilon} & I \end{array}$$

$$\begin{array}{ccc} I & \xrightarrow{0} & H \\ \downarrow{\scriptstyle \vee} & \searrow^{id} & \downarrow{\scriptstyle \varepsilon} \\ H & \xrightarrow{\varepsilon} & I \end{array}$$

Next we shall construct an n-cube $H(T)$ associated to every tree T with n internal edges. By virtue of its construction we shall have inclusion maps into this cube from $H(T')$ for every tree T' that can be made from T by contracting edges and these inclusions glue together nicely, in the sense that if T'' is a contraction of T' then the two inclusions $H(T'') \hookrightarrow H(T)$ and $H(T'') \hookrightarrow H(T') \hookrightarrow H(T)$ agree.

Definition 4.3.2 Let T be a planar tree with set of internal edges $E(T)$ of cardinality k. We shall assume that we have chosen a consistent convention for ordering $E(T)$. We define

$$H(T) = \bigotimes_{e \in E(T)} H.$$

Remark 4.3.3 Given a tree T, we can fix any ordering of internal edges at all for our consistent convention for ordering $E(T)$. The only reason for requiring it in the first place is that when we write products like $\bigotimes_{e \in E(T)} H$ where the order of factors is important, we want the i^{th} factor in the product to always corresponds to the same internal edge.

Remark 4.3.4 One should observe that the symmetries of T give $H(T)$ an automatic right $\mathrm{Aut}(T)$-action.

Definition 4.3.5 Let T be a planar tree and let D be a subset of its set of internal edges $E(T)$. We define

$$H_D(T) = \bigotimes_{e \in E(T)} H_e$$

where

$$H_e = \begin{cases} I & \text{if } e \in D. \\ H & \text{otherwise.} \end{cases}$$

We further define

$$H^-(T) = \bigcup_{D \neq \varnothing} H_D(T)$$

Remark 4.3.6 Let T/D be the tree obtained by contracting the edges in D. Then there is clearly a natural isomorphism

$$H_D(T) \xrightarrow{\sim} H(T/D).$$

This extends to an acyclic cofibration

$$H_D(T) \hookrightarrow H(T)$$

where we apply $0 : I \to H$ to H_e if $e \in D$ and the identity morphism otherwise. The pushout-product condition tells us that the induced map

$$H^-(T) \hookrightarrow H(T).$$

is an acyclic fibration.

The next two definitions concern arity 1 operations. Essentially these definitions allow us to ignore vertices with only one input edge during explicit computations. Informally, we just eliminate any such vertices by operadic composition.

Definition 4.3.7 If c is a nonempty set of unary (meaning having exactly one incoming edge) internal vertices of a tree T, there is a map

$$r_c : H(T) \to H(T/c)$$

where T/c is given by removing each vertex of c and connecting the incoming and outgoing edge. This map is given in terms of $\vee : H \otimes H \to H$ for vertices connecting two internal edges and $\varepsilon : H \to I$ for vertices connecting an internal and external edge.

Definition 4.3.8 Let $\mathbb{T}'$ be the set consisting of pairs (T, c), where $T \in \mathbb{T}$ and c is a nonempty set of unary internal vertices of T. We recursively define a function $\overline{\mathscr{P}}_c : \mathbb{T}' \to \mathbf{C}$ as follows. We fix $\overline{\mathscr{P}}_c(I_{\mathbb{T}}) = I$ and define

$$\overline{\mathscr{P}_c}(\gamma(t_n, T_1, \ldots, T_n)) = \begin{cases} I \otimes \overline{\mathscr{P}_c}(T_1) \otimes \cdots \otimes \overline{\mathscr{P}_c}(T_n) & \text{if } x \in c; \\ \mathscr{P}(n) \otimes \overline{\mathscr{P}_c}(T_1) \otimes \cdots \otimes \overline{\mathscr{P}_c}(T_n) & \text{otherwise.} \end{cases}$$

where x is the internal root vertex of t_n.

Remark 4.3.9 For all nonempty sets c, d of unary vertices in T with $c \subseteq d$, by the pushout-product condition the unit $u : I \to \mathscr{P}(1)$ induces a cofibration $\overline{\mathscr{P}_d}(T) \to \overline{\mathscr{P}_c}(T)$. Let us write (observing that $\overline{\mathscr{P}_\varnothing} = \overline{\mathscr{P}}(T)$)

$$\overline{\mathscr{P}_*}(T) = \bigcup_{c \neq \varnothing} \overline{\mathscr{P}_c}(T)$$

where c ranges over all the nonempty sets of unary vertices in T and the union is interpreted as the colimit over all cofibrations $\overline{\mathscr{P}_d}(T) \to \overline{\mathscr{P}_c}(T)$ for $c \subseteq d$. An application of pushout-product condition shows that the induced map

$$\overline{\mathscr{P}_*}(T) \to \overline{\mathscr{P}}(T)$$

is an $\mathrm{Aut}(\mathscr{P})$-cofibration for every well-pointed cofibrant operad $\mathscr{P}$.

We can now put these definitions together to create the W-construction. There are two distinct parts to this. First, we shall define the $\mathbb{S}$-module structure and then we shall define the operadic composition maps.

Definition 4.3.10 Let H be an interval. For any operad $\mathscr{P}$ we shall construct the operad $W(H, \mathscr{P})$ as the colimit of acyclic cofibrations of $\mathbb{S}$-modules.

$$W_0(H, \mathscr{P}) \hookrightarrow W_1(H, \mathscr{P}) \hookrightarrow W_2(H, \mathscr{P}) \hookrightarrow \cdots \tag{4.2}$$

The reader should be somewhat confused by this because we have not yet said what $W_i(H, P)$ is. Intuitively, $W_i(H, P)$ is the dimension $i - 2$ component of $W(H, P)$; the part corresponding to trees with at most i internal edges.

More formally, we construct $W_i(H, \mathscr{P})(n)$ by induction on i. For the base case, that is $i = 0$, we set

$$W_0(H, \mathscr{P})(n) := \mathscr{P}(n)$$

for all $n \geq 0$.

Now suppose that $i > 0$ and that $W_{i-1}(H, \mathscr{P})$ has been defined already. We shall also suppose that that we have a canonical map

$$\alpha_S : (H(S) \otimes \overline{\mathscr{P}}(S)) \otimes_{\mathrm{Aut}(S)} I[\mathbb{S}_n] \to W_{i-1}(H, \mathscr{P})(n)$$

for each tree S with at most $i-1$ internal edges and n input edges. (Recall that $\overline{\mathscr{P}}$ is the functor that we used in the construction of the free functor $\mathcal{F}$.) When $i=0$, these maps are

- When $n=1$, $\alpha_{I_{\mathbb{T}}}$ is the unit map $I \to \mathscr{P}(1)$
- When $n>1$, α_{t_n} is the identity map $\mathscr{P}(n) \to \mathscr{P}(n)$.

Notation 4.3.11 For notional simplicity we shall introduce the following shorthand

$$(H \otimes \mathscr{P})^-(T) := (H^-(T) \otimes \overline{\mathscr{P}}(T)) \cup_{H^-(T)\otimes\overline{\mathscr{P}_*}(T)} (H(T) \otimes \overline{\mathscr{P}_*}(T)).$$

Remark 4.3.12 By the pushout-product condition the inclusion map

$$(H \otimes \mathscr{P})^-(T) \hookrightarrow H(T) \otimes \overline{\mathscr{P}}(T)$$

is a cofibration.

Our next step shall be to use the maps α_S to construct a $\mathbb{S}_n$-equivariant map

$$\alpha_T^- : (H \otimes \overline{\mathscr{P}})^-(T) \otimes_{\mathrm{Aut}(S)} I[\mathbb{S}_n] \to W_{i-1}(H, \mathscr{P})(n). \tag{4.3}$$

Construction 4.3.13 (The construction of α_T^-) For every nonempty set D of internal edges define β_D as the composite

$$(H_D(T) \otimes \overline{\mathscr{P}}(T)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \to (H(T/D) \otimes \overline{\mathscr{P}}(T/D)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \xrightarrow{\alpha_{T/D}} W_{i-1}(H, \mathscr{P})(n)$$

where the first map is induced by both

- the isomorphism $H_D(T) \to H(T/D)$.
- the partial operad composition map $\overline{\mathscr{P}}(T) \to \overline{\mathscr{P}}(T/D)$.

Taking all choices of D we produce an $\mathbb{S}_n$-equivariant map

$$\beta_T : (H^-(T) \otimes \overline{\mathscr{P}}(T)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \to W_{i-1}(H, \mathscr{P})(n). \tag{4.4}$$

$$x \mapsto (\bigcup_{D\neq\varnothing} \beta_D)(x) \tag{4.5}$$

In an analogous manner, for each nonempty set c of internal unary vertices of T, the isomorphism $\overline{\mathscr{P}_c}(T) \to \overline{\mathscr{P}}(T/c)$ and the map $H(T) \to H(T/c)$ of Remark 4.3.7 together induce a map

$$\alpha_T^c : (H(T) \otimes \overline{\mathscr{P}_c}(T)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \to W_{i-1}(H, \mathscr{P})(n).$$

Once again these glue together over all possible nonempty choices of c to produce an $\mathbb{S}_n$-equivariant map

$$\delta_T : (H(T) \otimes \overline{\mathscr{P}_*}(T)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \to W_{i-1}(H, \mathscr{P})(n). \tag{4.6}$$

The maps β_T and δ_T glue together to give α_T^-.

Now we take the coproduct over all isomorphism classes of trees T with n input edges and i internal edges, and construct the following pushout.

$$\begin{array}{ccc}
\coprod_{[T],T\in\mathbb{T}(n,i)}(H\otimes\mathscr{P})^-(T)\otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] & \xrightarrow{\coprod\alpha_T^-} & W_{i-1}(H,\mathscr{P})(n) \\
\downarrow & & \downarrow \\
\coprod_{[T],T\in\mathbb{T}(n,i)}(H(T)\otimes\overline{\mathscr{P}}(T))\otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] & \xrightarrow{\coprod\alpha_T} & W_i(H,\mathscr{P})(n)
\end{array} \tag{4.7}$$

This defines both $W_i(H,\mathscr{P})$ and the maps α_T. Each inclusion $W_{i-1}(H,\mathscr{P})\to W_i(H,\mathscr{P})$ is a acyclic cofibration in the model structure on $\mathbb{S}_n$-modules, because it is the pushout of one.

Now we define $W(H,\mathscr{P})$ as in Definition 4.3.10 and one observes that $\mathscr{P}(n)=W_0(H,\mathscr{P})\to W(H,\mathscr{P})(n)$ is the composition of acyclic fibrations and so is one itself.

Remark 4.3.14 If we work with reduced operads, the construction above is somewhat simplified. In fact we are able to ignore all trees in $\mathbb{T}$ which have unary (one input edge) vertices. This is easy to see, because if T has a unary vertex then

$$\overline{\mathscr{P}_*}(T)=\overline{\mathscr{P}}(T).$$

It follows that

$$(H\otimes\mathscr{P})^-(T)=(H(T)\otimes\overline{\mathscr{P}}(T))\otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n].$$

The tree T thus contributes no extra structure to the pushout $W_i(H,\mathscr{P})(n)$, and so we lose nothing by omitting it.

Next we move on to describing the operad structure of $W(H,\mathscr{P})$.

Composition in $W(H,\mathscr{P})$ is induced by tree grafting in $\mathbb{T}$. Given a tree T with n input edges and n trees $T_1,\dots,T_n$ with k_i input edges respectively, one obtains a new tree $T'=\gamma(T,T_1,\dots,T_n)$ with $k=k_1+\cdots+k_n$ input edges. The n input edges of T become internal edges of T'. Therefore there is a map

$$H(T)\otimes H(T_1)\otimes\cdots\otimes H(T_n)\to H(T)\otimes H(T_1)\otimes\cdots\otimes H(T_n)\otimes I^n\to H(T').$$

where the first map is the canonical one and in the second map is induced by $1:I\to H$ on the newly created internal edges and the identity on everything else.

Notation 4.3.15 For notational simplicity, we shall use the subscript $\lambda[T_i]$ to represent tensoring by $-\otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n]$.

There is a map

$$\gamma : (H(T) \otimes \overline{\mathscr{P}}(T))_{\lambda[T]} \otimes \bigotimes_{i=1}^{n} (H(T_i) \otimes \overline{\mathscr{P}}(T_i))_{\lambda[T_i]} \to (H(T') \otimes \overline{\mathscr{P}}(T'))_{\lambda[T]}.$$

which is induced by composition in the free operad $\overline{\mathscr{P}}(T) \otimes \overline{\mathscr{P}}(T_1) \otimes \cdots \otimes \overline{\mathscr{P}}(T_n) \to \overline{\mathscr{P}}(T')$. The operad structure is determined by the unique requirement that the following diagram commutes

$$\begin{array}{ccc} (H(T) \otimes \overline{\mathscr{P}}(T))_{\lambda[T]} \otimes \bigotimes_{i=1}^{n} (H(T_i) \otimes \overline{\mathscr{P}}(T_i))_{\lambda[T_i]} & \longrightarrow & W(H, \mathscr{P})(n) \otimes \bigotimes_{i=1}^{n} W(H, \mathscr{P})(k_i) \\ \downarrow \gamma & & \downarrow \\ (H(T') \otimes \overline{\mathscr{P}}(T'))_{\lambda[T]} & \xrightarrow{\alpha_{T'}} & W(H, \mathscr{P})(n). \end{array}$$

Theorem 4.3.16 *Let* **C** *be a cofibrantly generated monoidal model category with a cofibrant unit I and an interval H. Let $\mathscr{P}$ be a $\mathbb{S}$-cofibrant well-pointed operad. Then the counit of the adjunction between pointed $\mathbb{S}$-modules and operads admits a factorisation*

$$\mathcal{F}_*(\mathscr{P}) \hookrightarrow W(H, \mathscr{P}) \xrightarrow{\sim} \mathscr{P}$$

into a cofibration f followed by a weak equivalence g. In particular, $W(H, \mathscr{P})$ is a cofibrant resolution for $\mathscr{P}$.

The proof of this result will depend on the technical Lemma 4.3.18. This lemma concerns the lifting properties of k-homomorphisms, a concept which we now define.

Firstly let T_1 and T_2 be two trees of arity k_1 and k_2, and with n_1 and n_2 internal edges respectively. Choose an integer $0 \leq i \leq k_1$ and consider $T = T_1 \circ_i T_2$. Recall that this is tree obtained by grafting T_2 onto T_1 at the i^{th} input edges. The tree T has internal edges coming from T_1, from T_2 and an extra input edge located at the grafting. There is a map

$$H(T_1) \otimes H(T_2) \to H(T)$$

where we apply $1 : I \to H$ to obtain the copy of H coming from this new internal edge. It follows from the definition of $\mathscr{P}$ that there is an isomorphism $\overline{\mathscr{P}}(T_1) \otimes \overline{\mathscr{P}}(T_2) \to \overline{\mathscr{P}}(T)$. These glue into a natural map

$$\gamma_e : \bigotimes_{i=1}^{2} (H(T_i) \otimes \overline{\mathscr{P}}(T_i)) \otimes_{Aut(T_i)} I[\mathbb{S}_{n_i}] \to (H(T) \otimes \overline{\mathscr{P}}(T)) \otimes_{Aut(T)} I[\mathbb{S}_{n_1+n_2-1}]$$

Secondly, observe that there is a map

$$\coprod_{j=1}^{n_1} \mathscr{P}(n_1) \otimes \mathscr{P}(n_2) \xrightarrow{\circ} \mathscr{P}(n_1 + n_2 - 1)$$

which for each j is given by the operadic partial composition map $\mathscr{P}(n_1) \otimes \mathscr{P}(n_2) \xrightarrow{\circ_j} \mathscr{P}(n_1+n_2-1)$.

Definition 4.3.17 Let H be an interval and let $\mathscr{P}$ and $\mathscr{Q}$ be operads. A *0-homomorphism* is a morphism of pointed $\mathbb{S}$-modules

$$\theta_0 : W_0(H,\mathscr{P}) \to \mathscr{Q}.$$

For $n > 0$, an *n-homomorphism* is a morphism of pointed $\mathbb{S}$-modules $\theta_n : W_n(H,\mathscr{P}) \to \mathscr{Q}$ with the following properties.

- The restriction of θ_n to $W_k(H,\mathscr{P})$, for $k < n$, is a k-homomorphism.
- For all $T = T_1 \circ_i T_2$ as directly above, the following diagram is commutative.

$$\begin{array}{ccc}
\bigotimes_{i=1}^{2}(H(T_i)\otimes\overline{\mathscr{P}}(T_i))\otimes_{Aut(T_i)} I[\mathbb{S}_{n_i}] & \xrightarrow{\gamma_e} & (H(T)\otimes\overline{\mathscr{P}}(T))\otimes_{Aut(T)} I[\mathbb{S}_{n_1+n_2-1}] \\
\Big\downarrow{\scriptstyle \alpha_{T_1}\otimes\alpha_{T_2}} & & \Big\downarrow{\scriptstyle \alpha_T} \\
\bigsqcup_{i=1}^{n_1} W_{k_1}(H,\mathscr{P})\otimes W_{k_2}(H,\mathscr{P}) & & W_k(H,\mathscr{P}) \\
\Big\downarrow{\scriptstyle \theta_1\otimes\theta_2} & & \Big\downarrow{\scriptstyle \theta_{k_1+k_2+1}} \\
\bigsqcup_{i=1}^{n_1} \mathscr{Q}(n_1)\otimes\mathscr{Q}(n_2) & \xrightarrow{\quad\circ\quad} & \mathscr{Q}
\end{array}$$

Lemma 4.3.18 *Let $k > 0$ and suppose the following square commutes,*

$$\begin{array}{ccc}
W_{k-1}(H,\mathscr{P}) & \xrightarrow{\theta_{k-1}} & \mathscr{Q} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \chi} \\
W(H,\mathscr{P}) & \xrightarrow{\omega} & \mathscr{R}
\end{array}$$

where ω is an operad morphism, χ is an operadic acyclic fibration, and θ_{k-1} is a $(k-1)$-homomorphism. Then there is a k-homomorphism θ_k, which when restricted to $W_{k-1}(H,\mathscr{P})$ is θ_{k-1}, and such that the following diagram commutes:

$$\begin{array}{ccc}
W_{k}(H,\mathscr{P}) & \xrightarrow{\theta_{k}} & \mathscr{Q} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \chi} \\
W(H,\mathscr{P}) & \xrightarrow{\omega} & \mathscr{R}.
\end{array}$$

Proof Let T be a tree with n input edges and k internal vertices. Let D be a set of internal edges of T. For each internal edge $e \in E(T)$, first we set

$$H_e^+ := \begin{cases} I \sqcup I & \text{if } e \in D. \\ H & \text{if } e \notin D. \end{cases}$$

and then define

$$H_D^+(T) := \bigotimes_e H_e^+.$$

Observe that the maps $I \overset{0}{\hookrightarrow} I \sqcup I \overset{(0,1)}{\hookrightarrow} H$ and the identity map $H \to H$ induce a sequence of cofibrations

$$H_D(T) \hookrightarrow H_D^+(T) \hookrightarrow H(T). \tag{4.8}$$

Let

$$H^+(T) := \bigcup_{D \neq \emptyset} H_D^+(T).$$

The morphisms in Diagram 4.8 glue to produce following sequence

$$H^-(T) \hookrightarrow H^+(T) \hookrightarrow H(T). \tag{4.9}$$

The maps in the sequence are cofibrations by the the pushout-product condition. Because we have fixed a convention for ordering the internal edges of T (Definition 4.3.2), both of these morphisms are equivariant under the natural action of $Aut(T)$. Next we define

$$(H(T) \otimes \overline{\mathscr{P}}(T))^+ := (H(T)^+ \otimes \overline{\mathscr{P}}(T)) \cup_{(H(T) \otimes \overline{\mathscr{P}}_*(T))} (H(T) \otimes \overline{\mathscr{P}}(T))$$

One thus has $Aut(T)$-equivariant cofibrations

$$(H(T) \otimes \overline{\mathscr{P}}(T))^- \hookrightarrow (H(T) \otimes \overline{\mathscr{P}}(T))^+ \hookrightarrow H(T) \otimes \overline{\mathscr{P}}(T)$$

In other words, we have factorized the morphism $(H(T) \otimes \overline{\mathscr{P}}(T))^- \hookrightarrow H(T) \otimes \overline{\mathscr{P}}(T)$. Now we push both maps out along the morphism of $\sqcup \alpha_T^-$. This will be a factorization of Diagram 4.7. Explicitly, we have the following diagram

$$\begin{array}{ccc}
\bigsqcup_{[T], T \in \mathbb{T}(n,k} (H \otimes \mathscr{P})^-(T) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] & \xrightarrow{\sqcup \alpha_T^-} & W_{k-1}(H, \mathscr{P})(n) \\
\downarrow & & \downarrow \\
(H(T) \otimes \overline{\mathscr{P}}(T))^+ \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] & \xrightarrow{\sqcup \alpha_T^+} & W_{k-1}^+(H, \mathscr{P})(n) \\
\downarrow & & \downarrow \\
\bigsqcup_{[T], T \in \mathbb{T}(n,k)} (H(T) \otimes \overline{\mathscr{P}}(T)) \otimes_{\mathrm{Aut}(T)} I[\mathbb{S}_n] & \xrightarrow{\sqcup \alpha_T} & W_k(H, \mathscr{P})(n)
\end{array} \tag{4.10}$$

We see that we have factorized the canonical morphism $W_{k-1}(H, \mathscr{P})(n) \hookrightarrow W_i(H, \mathscr{P})(n)$ as follows.

$$\begin{array}{ccc}
W_{k-1}(H, \mathscr{P}) & \hookrightarrow & W_k(H, \mathscr{P}) \\
\downarrow & \nearrow & \\
W_{k-1}^+(H, \mathscr{P}) & &
\end{array}$$

By the universal property of pushouts, any $(k-1)$-homomorphism $\theta_{k-1} : W_{k-1}(H, \mathscr{P}) \to Y$ can be uniquely extended to a map $\theta^+_{k-1} : W^+_{k-1}(H, \mathscr{P}) \to Y$ which satisfies the conditions of a k-homomorphism with respect to the maps α^+_T. Thus we have the following commutative diagram

$$\begin{array}{ccc} W^+_{k-1}(H, \mathscr{P}) & \xrightarrow{\theta^+_{k-1}} & \mathscr{Q} \\ \big\downarrow & \nearrow & \big\downarrow \chi \\ W_k(H, \mathscr{P}) & \xrightarrow{\omega} & \mathscr{R} \end{array} \tag{4.11}$$

Recall that χ is an acyclic fibration and therefore the dotted lift exists in the above diagram. This is the desired k-homomorphism θ_k. □

Proof (Theorem 4.3.16) First we shall show that morphism g appearing in the statement of the theorem is a weak equivalence. Let T be a tree with n input edges. The map ε from Definition 4.3.1 induces a morphism

$$H(T) \to I.$$

The operad structure of $\mathscr{P}$ induces a morphism $F : \overline{\mathscr{P}}(T) \to \mathscr{P}(n)$ as follows. If T is $I_{\mathbb{T}}$ then $F(T) = 1 \in \mathscr{P}(1)$, and if $T = t_n$ then F is the identity map. Otherwise if $T = \gamma(t_r, T'_1, \ldots, T'_r)$ we have then $F(T)$ is

$$F(T) = \gamma : \mathscr{P}(r) \otimes T'_1 \otimes \cdots \otimes T'_r \to \mathscr{P}(n).$$

These maps glue together to produce a map of operads $W(H, \mathscr{P}) \to \mathscr{P}$. In particular we observe that the composition

$$W_0(H, \mathscr{P}) \xrightarrow{p} W(H, \mathscr{P}) \xrightarrow{g} \mathscr{P}$$

is the identity. We earlier observed that the maps $W_i(H, \mathscr{P}) \to W_{i+1}(H, \mathscr{P})$ are acyclic cofibrations of collections and thus p is as well. Thus by the two out of three axiom, g is a weak equivalence.

Secondly, we study f in Theorem 4.3.16. This map is the morphism $\overline{\mathscr{P}} \hookrightarrow \overline{H(T) \otimes (\mathscr{P})}$ induced by $1 : I \to H$. We wish to show that this is a cofibration, a fact we are going to prove directly. Thus we must show that in any commutative square of operads

$$\begin{array}{ccc} \mathcal{F}_*(\mathscr{P}) & \longrightarrow & \mathscr{Q} \\ \big\downarrow f & \nearrow & \big\downarrow \chi \\ W(H, \mathscr{P}) & \xrightarrow{\omega} & \mathscr{R} \end{array}$$

where χ is an acyclic fibration, the dotted lift exists. By the universal property of the free operad, an operad map $\mathcal{F}_*(\mathscr{P}) \to \mathscr{S}$ is associated to a morphism of $\mathbb{S}$-modules $W_0(H, \mathscr{P}) = \mathscr{P} \to \mathscr{S}$. Therefore the above commutative square corresponds to a commutative square of $\mathbb{S}$-modules

$$\begin{array}{ccc} W_0(H,\mathscr{P}) & \xrightarrow{\theta_0} & \mathscr{Q} \\ \downarrow & \nearrow & \downarrow \chi \\ W(H,\mathscr{P}) & \xrightarrow{\omega} & \mathscr{R}. \end{array}$$

A dotted lift exists for the former square exists if and only a dotted lift exists for the latter.

From Lemma 4.3.18 and induction we have that for all $k > 0$ we have a map θ_k such that

$$\begin{array}{ccc} W_k(H,\mathscr{P}) & \xrightarrow{\theta_k} & \mathscr{Q} \\ \downarrow & \nearrow & \downarrow \chi \\ W(H,\mathscr{P}) & \xrightarrow{\omega} & \mathscr{R}. \end{array}$$

commutes and the restriction of θ_k to $W_0(H, \mathscr{P})$ is θ_0. The map $\operatorname{colim}_{k\to\infty} \theta_k$ thus provides the desired dotted lift.

Finally, we observe that since $\mathscr{P}$ is $\mathbb{S}$-cofibrant, by Lemma 4.2.8, $\mathcal{F}_*$ is a cofibrant operad. □

The last theoretical result of this chapter concerns the functoriality of the W-construction.

Proposition 4.3.19 *The construction $W(H, \mathscr{P})$ is functorial in both H and $\mathscr{P}$.*

The details of the proof of Proposition 4.3.19 may be found in [5].

4.4 Examples

This section describes what the W-construction looks like in practice, with a strong focus on the simplicial world. We start by discussing *associahedra* (sometimes called *Stasheff polytopes*); an infinite sequence of topological polytopes. These were originally discovered by Dov Tamari in his unpublished 1951 PhD thesis [31] (but later independently rediscovered by Jim Stasheff [30]), and significantly predate the study of model categories. Nonetheless the arity n component of the Boardman-Vogt resolution of the associative operad in topological spaces takes the form of a disjoint union of associahedra. We shall informally describe the calculation of these associahedra in low arities before stating some features of the general case. This section is primarily for intuition, as we shall then move on to studying Boardman-Vogt resolution of the associative operad in **simplicial sets**. Here, we shall do the low arity calculations by hand, before giving a complete combinatorial description. What we end up with is a simplicial set that can be realized geometrically as an associahedron (though we do not prove this). Finally, we shall treat the W-construction applied to the Barratt-Eccles operad.

4.4.1 Topological associahedra

Recall that in Top the unit is just the one point set $*$ and that the topological associative operad is defined by

$$\mathrm{Assoc}(n) := \bigsqcup_{\mathbb{S}_n} *.$$

In other words, it is just a discrete collection of points equipped with the free action of $\mathbb{S}_n$. As we are working in topological spaces, it is immediately apparent that this is $\mathbb{S}$-cofibrant and well-pointed. In Top one can verify that $[0,1]$, equipped with the maximum operation

$$\max : [0,1] \otimes [0,1] \to [0,1],$$

satisfies the axioms defining an interval. Thus $W([0,1], \mathrm{Assoc})$ will be a cofibrant replacement for Assoc.

Assoc is a reduced operad so by Remark 4.3.14 we do not need to consider trees with unary vertices. Also, $\mathrm{Assoc}(0) = \emptyset$, so if T has a vertex with no incoming edges $\overline{\mathrm{Assoc}} = \emptyset$, so we can ignore these trees as well. So when $n \geq 2$, the trees with n inputs that have the most internal edges are the binary trees with $n-2$ internal edges. Every other possible tree can be obtained from these via contraction.

Explicitly when $n = 1, 2$, the largest tree with n inputs has no internal edges so $W([0,1], \mathrm{Assoc})(1)$ is a point and $W([0,1], \mathrm{Assoc})(2)$ is a pair of points. The symmetric group acts on these in the obvious way.

When $n = 3$, we have three trees to consider. Two of these are binary trees and we also have t_3. The two binary trees have 1 internal edge and thus produce intervals. The tree t_3 which produces a point which is identified with 0 of both intervals. Thus it glues both intervals together. In fact, there will be 6 such intervals and the action of the symmetric group is just to permute them.

When $n = 4$ we have 5 binary trees with 2 internal vertices. These all produce $[0,1]^2$. These connect along 5 trees with 1 internal vertices, which means that these glue along a lines. Finally these glue along a single tree t_4 with no internal vertices. The final result is pentagon. Of course, there are $4! = 24$ such pentagons and the group $\mathbb{S}_4$ acts by permuting them.

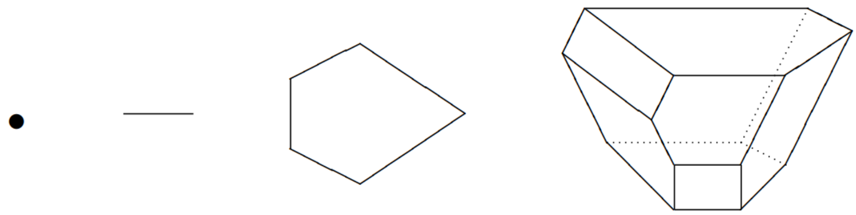


Figure 4.3: The small associahedra (image credit to [18])

In higher dimension $W([0,1], \text{Assoc})(n)$ will consist of $n!$ disjoint $n-2$ dimensional polytopes. These will each have

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

vertices, as this is the number of binary trees with n vertices. This polytope is known as the n^{th} associahedra $\mathcal{K}_n$. Figure 4.3 displays $\mathcal{K}_n$ for $n = 2, 3, 4, 5$.

To describe the operad structure on $W([0,1], \text{Assoc})$, select a point x within the polytope $\mathcal{K}_n$ and one point x_i each in the polytopes $\mathcal{K}_{k_i}$ for $k_i \in \mathbb{N}$ and $0 \leq i \leq n$. Then x has coordinates $(j_1, \dots j_{n-2})$ the $(n-2)$-cube $H(T)$ for some binary tree T and the x_i will have coordinates $(j_{i,1}, \dots j_{i,k_i-2})$ the $(n-2)$-cube $H(T_i)$ for some binary tree T_i with k_i leaves. Consider the tree $S = \gamma_{\mathbb{T}}(T, T_1, \dots, T_n)$. Then let y is the point

$$(j_1, \dots j_{n-2}, j_{1,1}, \dots j_{1,k_i-2}, \dots j_{n,k_n-2}, 1, \dots, 1) \in H(S)$$

Here we have adopted the convention that the first n coordinates correspond to the internal edges of S that come from T, the next k_1 are those coming from T_1 and so on. The final n points come from the newly created n internal edges of S which occur where the T_i have been grafted onto T. We have set all of these values equal to 1. Finally we recall that $H(S)$ has a canonical embedding into $\mathcal{K}_{k_1+\cdots+k_n}$. We thus have that $\gamma(x, x_1, \dots, x_n)$ is the image of y under this embedding.

4.4.2 Simplicial associahedra

In the category of simplicial sets, $\triangle^0$ is the initial object and the monoidal product is $\times$. Our first step shall be to show that $\triangle^1$ is an interval in $\text{Set}_\triangle$.

Lemma 4.4.1 *The standard* 1*-simplex is an interval in* $\text{Set}_\triangle$.

Proof There is a diagram of the form

$$\triangle^0 \sqcup \triangle^0 \hookrightarrow \triangle^1 \xrightarrow{\sim} \triangle^0.$$

The weak equivalence is just the terminal morphism. The cofibration is given by the pair $(0, 1)$. The map 0 corresponds to the morphism given on nondegenerate simplices by

$$\{0\} \mapsto \{0\}$$

and the map 1 corresponds to

$$\{0\} \mapsto \{1\}.$$

The morphism $\vee : \triangle^1 \times \triangle^1 \to \triangle^1$ is the maximum operation (ie. the operation induced by the maximum operation on the set $[1]$). It is easy to check that that diagrams appearing in definition 4.3.1 commute. □

Definition 4.4.2 The *associative operad in simplicial sets* is given in arity n by

$$\mathrm{Assoc}(n) := \coprod_{\mathbb{S}_n} \triangle^0.$$

equipped with the free action of $\mathbb{S}_n$.

The very low arity calculations are trivial.

Example 4.4.3 In arity 0,1 and 2, there are no trees to worry about other than the trivial one, the corolla. It follows immediately that

$$W(\triangle^1, \mathrm{Assoc})(n) = W_0(\triangle^1, \mathrm{Assoc})(n) = \coprod_{\mathbb{S}_n} \triangle^0.$$

for $n = 0, 1, 2$.

Remark 4.4.4 One can easily check that

$$H(T) \times \overline{\mathrm{Assoc}}(T) \times_{Aut(T)} I[S_n] \cong H(T) \times \mathrm{Assoc}(n).$$

This immediately implies that

$$W(\triangle^1, \mathrm{Assoc})(n) = \mathcal{K}_n \times \mathrm{Assoc}(n)$$

for some simplicial set $\mathcal{K}_n$. In other words $W(\triangle^1, \mathrm{Assoc})(n)$ consists of $n!$ copies of the *simplicial associahedron* $\mathcal{K}_n$. We have not yet seen any fully worked out examples of the W-construction. Therefore in the next two examples, we shall ignore this 'obvious' simplification in the interests of gaining fluency with the computations involved. We shall return to it in the proof of Theorem 4.4.11.

Example 4.4.5 The first nontrivial case is $n = 3$. To begin with we have

$$W_0(\triangle^1, \mathrm{Assoc})(3) = \coprod_{\mathbb{S}_3} \triangle^0.$$

Figure 4.4: The arity 3 trees with one internal edge

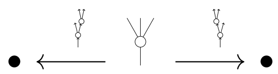


Figure 4.5: The simplicial set $W(\triangle^1, \mathrm{Assoc})(3)$

This comes equipped with a map

$$\alpha_{t_3} : (H(t_3) \times \overline{\mathrm{Assoc}}(t_3)) \times_{Aut(t_3)} I[\mathbb{S}_3] \to W_0(\triangle^1, \mathrm{Assoc})(3)$$

We observe that the domain and codomain of α_{t_3} are $\mathrm{Assoc}(3)$ and we recall from the previous section that α_{t_3} is defined to be the identity on $\mathrm{Assoc}(3)$. To compute $W_1(\triangle^1, \mathrm{Assoc})(3)$ we must consider the trees with one internal edge and three input edges. These are the two binary trees which are illustrated in figure 4.4. We shall first study the tree on the left, which we shall denote T. Because there is only one internal edge, we have that $H(T) = \triangle^1$ and $H(T)^- = \triangle^0$. The cofibration

$$H(T)^- \hookrightarrow H(T)$$

given by the map $0 : \triangle^0 \to \triangle^1$. There is a partial operad composition map

$$\circ_1 : \mathrm{Assoc}(2) \times \mathrm{Assoc}(2) \to \mathrm{Assoc}(3).$$

Combining these we obtain

$$\alpha_T^- : (H(T) \times \overline{\mathrm{Assoc}}(T))^- \times_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \to W_0(\triangle^1, \mathrm{Assoc})(3)$$

as $id_{\triangle^0} \times \circ_1$. We also have a map

$$r_T : (H(T) \times \overline{\mathrm{Assoc}}(T))^- \times_{\mathrm{Aut}(T)} I[\mathbb{S}_n] \to (H(T) \times \overline{\mathrm{Assoc}}(T)) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_n]$$

induced by the 0 map. We can repeat this analysis for the right tree S in Figure 4.4. The only difference is that the partial composition operation is now $\circ_2$. To finish calculating $W_1(\triangle^1, \mathrm{Assoc})(3)$ we now must calculate the pushout appearing in Equation (4.7). The result is the simplicial set appearing in Figure 4.5 tensored by $\coprod_{\mathbb{S}_3} \triangle^0$. The simplicial set in the figure consists of three 0-simplices, one of which is associated with the 3-corolla.

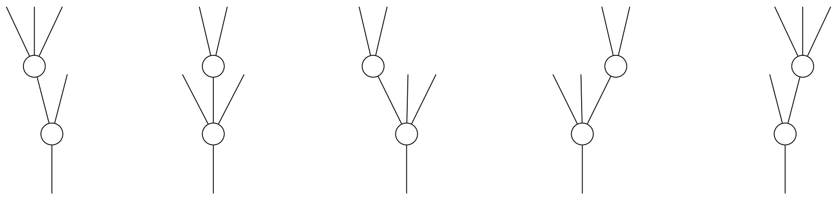


Figure 4.6: The arity 4 trees with one internal edge

There are two nondegenerate 1-simplices, one associated to S and the other to T. These have the property that if we apply d_0 to either of them, the result is the 0-simplex associated to t_3. The result of tensoring by $\bigsqcup_{\mathbb{S}_3} \triangle^0$ is that we have 6 disjoint copies of this simplex, each labelled with an element of $\mathbb{S}_3$.

Recall that the pushout also gives us maps

$$\alpha_T : (H(T) \times \overline{\text{Assoc}}(T)) \times_{\text{Aut}(T)} I[\mathbb{S}_n] \to W_0(\triangle^1, \text{Assoc})(3).$$

One observe that $(H(T) \times \overline{\text{Assoc}}(T)) \times_{\text{Aut}(T)} I[\mathbb{S}_n]$ is simply 6 copies of $H(T)$ indexed by $\mathbb{S}_3$. The map α_T consists of mapping each copy of $H(T)$ to 1-simplex on the right via the identity map in the appropriate (meaning labelled with the same element of $\mathbb{S}_3$) copy of Figure 4.5. There are no larger trees with three input vertices, and thus we conclude that $W_1(\triangle^1, \text{Assoc})(3)$ is also $W(\triangle^1, \text{Assoc})(3)$.

Example 4.4.6 Moving on to the $n = 4$ case, we have as before that

$$W_0(\triangle^1, \text{Assoc})(4) = \bigsqcup_{\mathbb{S}_4} \triangle^0.$$

This comes equipped with the identity map

$$\alpha_{t_4} : (H(t_4) \times \overline{\text{Assoc}}(t_4)) \times_{Aut(t_4)} I[\mathbb{S}_4] \to W_0(\triangle^1, \text{Assoc})(4).$$

There are five trees with four input vertices and one internal edge. These are shown in Figure 4.6. We can repeat the analysis appearing in the case $n = 3$ to build $W_1(\triangle^1, \text{Assoc})(4)$. This results in 4! copies of Figure 4.7, with these copies indexed by $\mathbb{S}_4$.

Finally, as we can see from Figure 4.8, there are five trees with four input edges and two internal edges. We shall study the tree T on the far left of the figure. We have labelled its internal edges by a and b. We shall denote the tree obtained by collapsing a by T_a (resp. b by T_b). We have that $H(T) = (\triangle^1)_a \times (\triangle^1)_b$ where here $(\triangle^1)_i$ means that this is the component of $H(T)$ associated to the internal edge x. $H(T_a) \cong H_a(T) = \triangle^1_b$ and $H(T_b) \cong H_b(T) = \triangle^1_a$ embed into $H(T)$ in the obvious way. Finally $H(t_4) = \triangle^0$ embeds into $H(T), H(T_a)$ and $H(T_b)$ via

$$\{0\} \mapsto \{0\}.$$

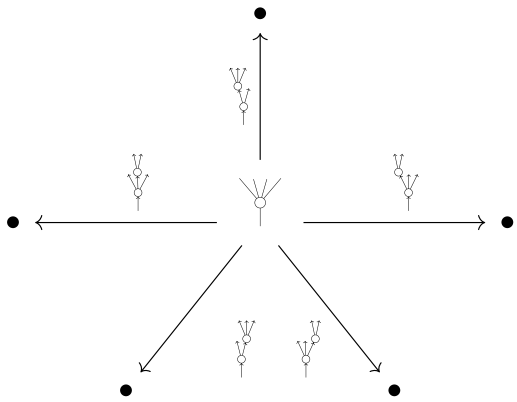


Figure 4.7: $W_1(\triangle^1, \text{Assoc})(4)$

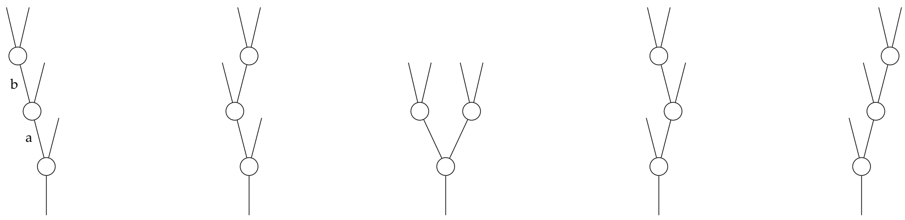


Figure 4.8: The arity 4 trees with two internal edges

The morphism

$$id_{\text{Assoc}(2)} \times \circ_1 : \text{Assoc}(2) \times \text{Assoc}(2) \times \text{Assoc}(2) \to \text{Assoc}(2) \times \text{Assoc}(3)$$

is the map $\overline{\text{Assoc}}(T) \to \overline{\text{Assoc}}(T_a)$ and $\circ_1 \times id_{\text{Assoc}(2)}$ is the map $\overline{\text{Assoc}}(T) \to \overline{\text{Assoc}}(T_b)$. These maps together describe the map

$$\alpha_T^- : ((\triangle_a^1 \sqcup \triangle_b^1)/\triangle^0) \times \text{Assoc}(4) \to W_1(\triangle^1, \text{Assoc})(4).$$

We also have the obvious embedding

$$r_T : ((\triangle_a^1 \sqcup \triangle_b^1)/\triangle^0) \times \text{Assoc}(4) \hookrightarrow \triangle_a^1 \times \triangle_b^1 \times \text{Assoc}(4).$$

Taking the pushout (as in Diagram 4.7), we obtain the part of $W_2(\triangle^1, \text{Assoc})(3)$ corresponding to T, as illustrated in Figure 4.9.

Doing this for all the trees in Figure 4.8 we obtain the Figure 4.10. In the diagram we have placed the label for each $H(T)$ at the terminal vertex (see Definition 4.4.10). There are no trees of arity 4 with more than two internal edges, so we conclude that $W(\triangle^1, \text{Assoc})(4)$ is 24 copies of Diagram 4.10, each indexed with an element of S_4.

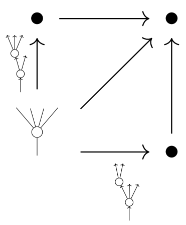


Figure 4.9: Part of $W_2(\triangle^1, \mathrm{Assoc})(3)$

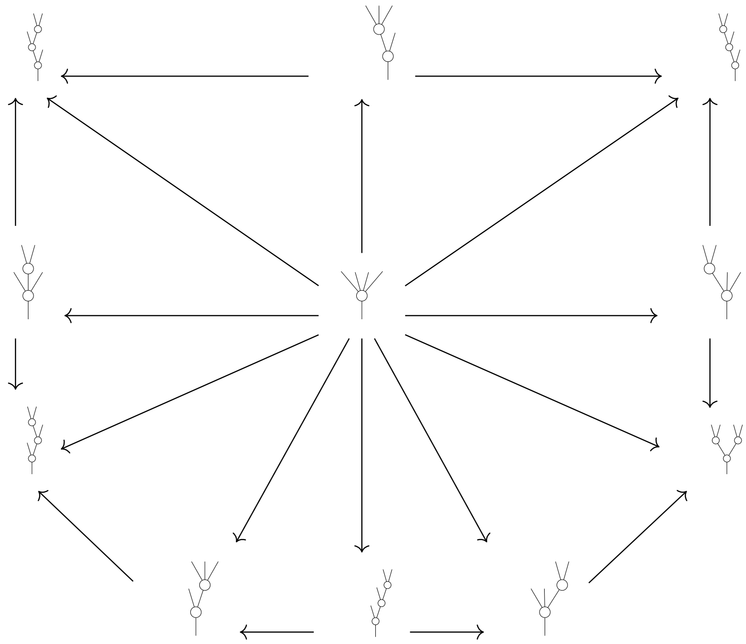


Figure 4.10: $W(\triangle^1, \mathrm{Assoc})(4)$

Definition 4.4.7 A directed graph $\mathcal{G}$ is called *transitively closed* it satisfies the condition that

$$(a,b),(b,c) \in E(\mathcal{G}) \implies (a,c) \in E(\mathcal{G}),$$

where $E(G)$ is the edge set of the graph. In other words, if there is a directed edge from vertices a to b and another from vertices b to c, then there is a edge from a to c.

A transitively closed graph can be regarded as a category.

Definition 4.4.8 Let $\mathcal{G}$ be a transitively closed graph. We define a category

$\overline{\mathcal{G}}$ with objects given by the vertices of $\mathcal{G}$ and with

$$\mathrm{Hom}(a,b) = \begin{cases} \{id\} & \text{if } a = b. \\ \emptyset & \text{if } a \neq b \text{ and } (a,b) \notin E(\mathcal{G}). \\ \{(a,b)\} & \text{if } a \neq b \text{ and } (a,b) \in E(\mathcal{G}). \end{cases}$$

The composite of the morphisms (a,b) and (b,c) is (a,c).

Before describing $W(\triangle^1, \mathrm{Assoc})(n)$, it will be necessary to understand the combinatorial structure of $(\triangle^1)^{\times n}$.

Lemma 4.4.9 *Let $\mathcal{I}$ be a directed graph on 2^n vertices, each labelled by a sequence of length n with elements in $\{0,1\}$, and such there exists an edge $v_{(\sigma_1,\sigma_2,\ldots,\sigma_n)}$ to $v_{(\tau_1,\tau_2,\ldots,\tau_n)}$ if and only if $\sigma_i \leq \tau_i$ for all i. Then this graph is transitively closed and*

$$(\triangle^1)^{\times n} = \mathcal{N}(\overline{\mathcal{I}})$$

Proof Each 0-simplex of $(\triangle^1)^{\times n}$ is of the form $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$ where $\sigma_i \in [1]$. There is a 1-simplex $\rho \in (\triangle^1)^{\times n}$ such that $d_0(\rho) = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$ and $d_1(\rho) = \tau_1 \times \tau_2 \times \cdots \times \tau_n$ if and only if $\sigma_i \leq \tau_i$ for all i. For $m \geq 3$ the non-degenerate simplices of $(\triangle^1)^{\times n}$ are

$$((\triangle^1)^{\times n}_m)^{nd} = \{s_{(q_1)}(\theta_1) \times s_{(q_2)}(\theta_2) \times \cdots \times s_{(q_n)}(\theta_n) : m > q_i \geq 0 \text{ and } q_i \neq q_j \text{ when } i \neq j\}$$

where θ_i is the 1-simplex in the i^{th} copy of $\triangle^1$ in the product, where $s_{(k)} = s_{n-1}s_{n-2}\cdots\widehat{s_k}\cdots s_0$ and we use the notation $\widehat{x}$ to mean that we omit x. The nondegenerate simplices of $\mathcal{N}(\overline{\mathcal{I}})_m$ will be the simplices of the form

$$(\sigma_{1,1} \times \cdots \times \sigma_{1,n}) \circ \cdots \circ (\sigma_{m,1} \times \cdots \times \sigma_{m,n})$$

where

- $\sigma_{i,j} \in \triangle^1_1$.
- for each $i \in [1,m]$, there exists a unique $p(i) \in [1,n]$ such that $\sigma_{i,p(i)} = \mathrm{id}$.
- fix $j \in [1,n]$ and suppose that there exists $q(j) \in [1,n]$ such that $\sigma_{q(j),j} = \mathrm{id}$. Then for all $i > q(j)$, $\sigma_{i,j} = s_0(1)$ and for all $i < q(j)$, $\sigma_{i,j} = s_0(0)$.

Then there is a function

$$f : \mathcal{N}(\overline{\mathcal{I}})^{ng}_m \to ((\triangle^1)^{\times n}_m)^{ng}$$

$$(\sigma_{1,1} \times \cdots \times \sigma_{1,n}) \circ \cdots \circ (\sigma_{m,1} \times \cdots \times \sigma_{m,n}) \mapsto s_{(p(1)-1)}(\mathrm{id}) \times \cdots \times s_{(p(n)-1)}(\mathrm{id})$$

One can check that this extends to an isomorphism of simplicial sets between $\mathcal{N}(\overline{\mathcal{I}})$ and $(\triangle^1)^{\times n}$. □

Definition 4.4.10 The *terminal vertex* of $(\triangle^1)^{\times n}$ is the vertex $v_{(1,1,\ldots,1)}$.

Theorem 4.4.11 *Let $\mathcal{G}$ be the directed graph on*

$$\frac{3P_{n-1}(3) - P_{n-2}(3)}{4n}$$

vertices, where P_n is the n^{th} Legendre polynomial and where each vertex v_T is labelled by an n-ary non-unital tree. The edges of $\mathcal{G}$ are defined as follows; there is a directed edge from v_T to v_S if and only there exists $D \subseteq E(S)$ such that $T = S/D$. Then this graph is transitively closed and

$$W(\triangle^1, \mathrm{Assoc})(n) = \coprod_{\mathbb{S}_n} \mathcal{N}(\overline{\mathcal{G}})$$

where the action of $\mathbb{S}_n$ given by permutation of components of the disjoint union.

Remark 4.4.12 Recall that the Legendre polynomials $P_n(x)$ are a system of complete and orthogonal polynomials, indexed by the positive integers and defined inductively by setting $P_1(x) = 1$ and requiring that $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ for all $m < n$. The sequence

$$a_n = \frac{3P_{n-1}(3) - P_{n-2}(3)}{4n}$$

is known as the little Schroeder numbers (OEIS sequence A001003 [15]). In particular a_n counts the number of non-unital n-ary trees, for a proof see [28].

Proof First, we make the observation that $H(T) \times \overline{\mathrm{Assoc}}(T)) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_n]$ is isomorphic to $H(T) \times \coprod_{\mathbb{S}_n} \triangle^0$. The simplicial set $W(\triangle^1, \mathrm{Assoc})(n)$ therefore has the form

$$\coprod_{\mathbb{S}_n} \mathcal{K}_n \qquad\qquad (\text{or } \mathcal{K}_n \times \mathrm{Assoc}(n))$$

where $\mathcal{K}_n$ is a simplicial set that remains to be computed. We shall do this by induction. Our inductive hypothesis shall be that $W_i(\triangle^1, \mathrm{Assoc})(n)$ admits the following description.

Let $\mathcal{H}_i$ be a directed graph with vertices indexed by the set of trees with n input edges and i or fewer internal edges, and such that there exists a directed edge from v_T to v_S if and only there exists $D \subseteq E(S)$ such that $T = S/D$. Then

$$W_i(\triangle^1, \mathrm{Assoc})(n) = \coprod_{\mathbb{S}_n} \mathcal{N}(\overline{\mathcal{H}_i})$$

Moreover we assume, letting $\mathcal{I}$ be as in Lemma 4.4.9, that the map α_T is $\mathcal{N}(\overline{\beta_T})$ where

$$\beta_T : \mathcal{I} \to \mathcal{H}_i$$

is the map which sends the vertex in $\mathcal{I}$ indexed by the binary sequence $(\sigma_1, \dots \sigma_k)$ to the vertex in $\mathcal{H}_i$ indexed by the tree T/F where $F = \{e_i \in E(T) : \sigma_i = 0\}$.

Firstly, when $i = 0$, we have that $W_0(\triangle^1, \text{Assoc})(n) = \text{Assoc}(n)$. So our hypothesis holds in the base case.

Secondly, recall that the following diagram, which is a specialization of 4.7, is a pushout

$$\begin{array}{ccc} \bigsqcup_{[T],T\in\mathbb{T}(n,k+1)} H^-(T) \times \text{Assoc}(n) & \xrightarrow{\sqcup \alpha_T^-} & W_k(\triangle^1, \text{Assoc})(n) \\ \downarrow & & \downarrow \\ \bigsqcup_{[T],T\in\mathbb{T}(n,k+1)} H(T) \times \text{Assoc}(n) & \xrightarrow{\sqcup \alpha_T} & W_{k+1}(\triangle^1, \text{Assoc})(n) \end{array}$$

Without loss of generality, we can ignore the symmetric action. This is now exactly what we wish to show because for each $T \in \mathbb{T}(n, k+1)$ we have

- The simplicial subset
$$\alpha_T (H(T)) \subset W_{k+1}(\triangle^1, \text{Assoc})(n)$$
has exactly one vertex, the terminal one, which is not in $W_k(\triangle^1, \text{Assoc})(n)$. We define this to be the vertex v_T.
- Let $\sigma \in W_{k+1}(\triangle^1, \text{Assoc})(n)$ be a 1-simplex. Then $d_1(\sigma) = v_T$ if and only if $d_0(\sigma) \in \alpha_T (H(T))$, that is, if $d_0(\sigma)$ is indexed by a tree S such that there exists $D \in E(T)$ such that $S = T/D$.
- By Lemma 4.4.9, $\alpha_T (H(T)) \cong (\triangle^1)^{\times k+1}$ is equal to $\mathcal{N}(\overline{\mathcal{I}})$. One can show that the nerve functor preserves small coproducts in the category of small categories. So
$$W_{k+1}(\triangle^1, \text{Assoc})(n) = \mathcal{N}(\mathcal{H}_{k+1})$$
as desired.

Therefore, our inductive hypothesis holds for $i = k + 1$

Finally, as noted in Remark 4.3.14, the construction of $W(\triangle^1, \text{Assoc})(n)$ relies only on n-ary trees with no unitary vertices. Therefore our induction terminates once we reach $W_{n-2}(\triangle^1, \text{Assoc})(n)$. □

Remark 4.4.13 The operadic composition morphisms are exactly as in the topological case in the previous subsection.

4.4.3 The homotopy Barratt-Eccles E_n-operad

This subsection is dedicated to giving a concrete description of the Boardman-Vogt resolution of the Barratt-Eccles E_n-operad. The 0-simplices of the Barratt-Eccles operad are the same as those of the associative operad, so some of our analysis in the last subsection carries over. The new feature is that we now have non-degenerate simplices in dimension greater than 0. These behave in a more complicated fashion. More precisely, as we saw above, each tree in $\mathbb{T}$ with k internal vertices is associated to $n!$ copies of the k-cube $H(T)$ in the associahedron, indexed by $\mathbb{S}_n$. We are also to show that there is a subset K_T of $\mathbb{S}_n$ associated to T. Every i-simplex σ in $\Gamma^{(k)}(n)$ has $i+1$ vertices, which are elements of $\mathbb{S}_n$. If all these vertices are in K_T, then $\sigma \times H(T)$ is a simplex in $W(\triangle^1, \Gamma^{(k)})(n)$. Let us begin.

Definition 4.4.14 Let T be a tree with n input edges and k internal edges. Let T' be a tree given by collapsing one of its internal edges d. Then there is a group homomorphism

$$f_d : Aut(T) \to Aut(T')$$

given by partial composition in the associative operad at the collapsed edge d. Since $Aut(t_n) = \mathbb{S}_n$, iterating this procedure until we arrive at the n-corolla induces a map

$$F_T : Aut(T) \to \mathbb{S}_n.$$

One can easily check that this map is independent of the order in which we collapse the internal edges. The image of F_T will be a subset of $\mathbb{S}_n$ which we call the *subset associated to T*.

Example 4.4.15 Consider the left tree in Figure 4.4. We know that $Aut(T) = \mathbb{S}_2 \times \mathbb{S}_2$ and the map F_T is given by the map $\circ_1 : \mathbb{S}_2 \times \mathbb{S}_2 \to \mathbb{S}_3$. The subset of $\mathbb{S}_3$ associated to T will be the permutations

$$\{e, (1,2), (1,2,3), (1,3)\}$$

Similarly the relevant map for the right tree in the figure would be $\circ_2$. The subset of $\mathbb{S}_3$ associated to this tree will be the permutations

$$\{e, (1,3), (1,3,2), (2,3)\}$$

We shall can now give our description of the Boardman-Vogt resolution of the Barratt-Eccles E_n-operad $W(\triangle^1, \Gamma^{(n)})$.

Notation 4.4.16 Consider $W(\triangle^1, \text{Assoc})$ as described in the previous section. Recall that, for each $T \in \mathbb{T}(r)$ there is a map

$$\alpha_T : H(T) \times \overline{\text{Assoc}}(T)) \times_{\text{Aut}(T)} I[\mathbb{S}_r] \to W_r(\triangle^1, \text{Assoc})(r) \hookrightarrow W(\triangle^1, \text{Assoc})(r)..$$

Further recall that $H(T) \times \overline{\text{Assoc}}(T)) \times_{\text{Aut}(T)} I[\mathbb{S}_r] \cong \mathbb{S}_r$. Under this identification, for each $\sigma \in \mathbb{S}_r$ we define

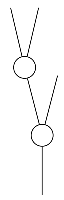
$$H(T)_\sigma := \alpha_T\,(H(T) \times \sigma) \subset W(\triangle^1, \text{Assoc})(r).$$

Theorem 4.4.17 *Let $n > 0$ be an integer. The Boardman-Vogt resolution of the Barratt-Eccles E_n-operad $W(\triangle^1, \Gamma^{(n)})$ admits the following complete description.*

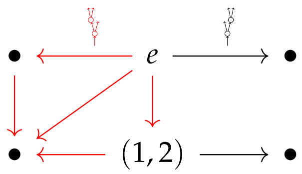
- *The simplicial set $W(\triangle^1, \Gamma^{(n)})$ has a simplicial subset isomorphic to $W(\triangle^1, \mathrm{Assoc})$. This subset contains the entire 0-skeleton of $W(\triangle^1, \Gamma^{(n)})$.*
- *Let $\sigma = (\sigma_1, \ldots, \sigma_k) \in \Gamma^{(n)}$. Let $T \in \mathbb{T}$ such that $\sigma_i \in K_T$, for $0 \leq i \leq k$. Then $G_T^\sigma = \sigma \times H(T)$ is a simplicial subset of $W(\triangle^1, \Gamma^{(n)})$ such that $\sigma_i \times H(T) = H(T)_{\sigma_i}$ for $0 \leq i \leq k$.*
- *Every simplex of $W_i(\triangle^1, \Gamma^{(n)})(r)$ is in either $W(\triangle^1, \mathrm{Assoc})$ or one of the G_T^σ.*

Before we prove this result, we shall briefly illustrate what it means in practice.

Example 4.4.18 Consider the simplex $\sigma = (e, (1,2)) \in \Gamma(3)_1$. The permutations e and $(1,2)$ are in K_T where T is the tree



The simplicial subset G_T^σ is therefore the red part of the following diagram.



where the top line is the associahedron with index e and the second line is the associahedron with index $(1,2)$.

Proof By Remark 4.3.14, the construction of $W(\triangle^1, \Gamma^{(n)})(r)$ relies only on r-ary trees with no unitary vertices. We shall use induction on the number i of internal edges of these trees. To be precise, we shall assume that $W_i(\triangle^1, \Gamma^{(n)})(r)$ admits the following description.

- The simplicial set $W_i(\triangle^1, \Gamma^{(n)})(r)$ has a simplicial subset isomorphic to $W_i(\triangle^1, \mathrm{Assoc})(r)$. This subset contains the entire 0-skeleton of $W_i(\triangle^1, \Gamma^{(n)})(r)$.
- Let $\sigma = (\sigma_1, \ldots, \sigma_k) \in \Gamma^{(n)}$. Let $T \in \mathbb{T}(r, i)$ such that $\sigma_j \in K_T$, for $0 \leq j \leq k$. Then $G_T^\sigma = H(T) \times \sigma$ is a simplicial subset of $W(\triangle^1, \Gamma^{(n)})$ such that $H(T) \times \sigma_j = H(T)_{\sigma_j}$ for $0 \leq j \leq k$. Moreover, the map α_T is induced by the identity map $H(T) \times \sigma \to G_T^\sigma$ for all $\sigma \in \overline{\Gamma^{(n)}}(T)$.

- Every simplex of $W(\triangle^1, \Gamma^{(n)})$ is in either $W(\triangle^1, \mathrm{Assoc})$ or one of the G_T^σ.

First observe that

$$W_0(\triangle^1, \Gamma)(r) := \Gamma(r).$$

Since $K_{t_r} = \mathbb{S}_r$, our hypothesis is true when $i = 0$. Next, suppose that it is true when $i = k$. Then

$$\begin{array}{ccc}
\bigsqcup_{[T], T \in \mathbb{T}(r,k+1)} (H \times \Gamma^{(n)})^-(T) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_r] & \xrightarrow{\bigsqcup \alpha_T^-} & W_k(\triangle^1, \Gamma^{(n)})(r) \\
\downarrow & & \downarrow \\
\bigsqcup_{[T], T \in \mathbb{T}(r,k+1)} (H(T) \times \overline{\Gamma^{(n)}}(T)) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_r] & \xrightarrow{\bigsqcup \alpha_T} & W_{k+1}(\triangle^1, \Gamma^{(n)})(r)
\end{array}$$

Let $\sigma = (\sigma_0, \ldots, \sigma_k) \in \Gamma^{(n)}$. Then one can easily show that $\sigma \in \overline{\Gamma^{(n)}}(T)$ if and only if $\sigma_i \in K_T$ for $0 \leq i \leq k$. Therefore $\bigsqcup_{[T], T \in \mathbb{T}(r,k+1)} (H(T) \times \overline{\Gamma^{(n)}}(T)) \times_{\mathrm{Aut}(T)} I[\mathbb{S}_r]$ is the union of two kinds of simplicial sets.

- We have a $(k-1)$-cube $H(T) \times (\sigma_0)$ for each tree $T \in \mathbb{T}(n, k+1)$ and each 0-simplex $\sigma_0 \in \Gamma^{(n)}$.
- We have a simplicial set $H(T) \times \sigma$ for each $\sigma \in \Gamma^{(n)}$ such that the vertices of σ lie in K_T.

By our inductive hypothesis, the simplicial subset

$$\left(\bigsqcup_{[T], T \in \mathbb{T}(r,k+1)} \alpha_T \right) \left(\bigcup_{\sigma_0 \in \mathbb{S}_r} H(T) \times (\sigma_0) \right) \subseteq W_{k+1}(\triangle^1, \Gamma^{(n)})(r)$$

will be isomorphic to $W_{k+1}(\triangle^1, \mathrm{Assoc})(r)$. The second type of simplicial subset are exactly the subsets of type 2. $\square$

Chapter 5

Simplicial coalgebras

The purpose of this chapter is to generalise Theorem 2.4.10, which says that n-fold suspensions are coalgebras over the little n-discs operad, to simplicial sets. There are numerous reasons why this is a useful idea. Our initial motivation was to find a strictly coassociative topological coalgebra, to provide the Eckmann-Hilton dual of the Moore loop space. Unfortunately, for unclear reasons, we have failed to find any such model in either topological spaces or simplicial sets. Another possible motivation therefore is that passing through simplicial sets is a necessary step on the journey to describing E_n-coalgebras in the category of ∞-operads. Among other things, this may permit a dualization of Lurie's proof of May's recognition principle [21, Theorem 5.2.6.15].

Our discussion shall proceed as follows. In the first section, we shall shall define the 'correct' notion of a coendomorphism operad in topological spaces. In the second section we will show that simplicial n-fold suspensions are homotopy algebras over the Barratt-Eccles operad, the main result of this thesis.

5.1 The simplicial coendomorphism operad

In this chapter, we wish to extend the notion of coalgebras to the realm of simplicial sets. As Moreno-Fernández and Wierstra did in topological spaces (see Chapter 1), we are going to do this by defining the notion of a coendomorphism operad. This is significantly more difficult than it appears. We cannot simply take the obvious choice, the operad defined in arity n by

$$\mathrm{Map}_{\mathrm{Set}_{\triangle}}(X, X^{\vee n}).$$

To see why, consider what happens in the case where $X = S^1$. This simplicial set has only 1 nondegenerate simplex other than the base point - the 1-

simplex σ. The vertices of the simplicial set

$$\mathrm{Map}_{\mathrm{Set}_\triangle}(S^1, (S^1)^{\vee n})$$

are distinctly non-interesting, because the σ can only mapped to one copy of S^1 in the wedge product. This is in total contrast with the interesting structure in topological spaces, and hence in the homotopy category.

This hints at the underlying problem. As we have seen throughout this report, not all simplicial sets are fibrant in the Kan-Quillen model structure. Thus, not all maps in the homotopy category exist between all pairs of objects in the model. To ensure that they do we must take a fibrant replacement of $X^{\vee n}$. (We shall later see that $\mathrm{Map}_{\mathrm{Set}_\triangle}(X, X^{\vee n})$ is not even the correct homotopy type to be a good candidate for a coendomorphism operad.) To ensure things remain as combinatorially tractable as possible, we shall use Kan's Ex^∞ functor for this task (we could alternatively use $S_\bullet|X|$, and we shall actually use this approach in the proof of Theorem 5.2.1). The underlying $\mathbb{S}$-module of the desired operad is very easy to describe and we can do this immediately.

Definition 5.1.1 We define the *simplicial coendomorphism* $\mathbb{S}$*-module* in arity r to be

$$\mathrm{CoEnd}(X)(r) := \mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(X^{\vee r})).$$

Each $\sigma \in \mathbb{S}_r$ induces a map $\sigma^* : X^{\vee r} \to X^{\vee r}$, by via permutation of the factors of the wedge product. Then the symmetric action of the $\mathbb{S}$-module is given by the maps

$$- * \sigma : \mathrm{CoEnd}(X)(r) \to \mathrm{CoEnd}(X)(r)$$
$$f \mapsto \sigma^* \circ f.$$

Remark 5.1.2 It is obvious that $- * \sigma$ is a bona fide simplicial map because the degeneracy and face maps of the simplicial mapping space act only on the domain of a n-simplex $f : X \times \triangle^m \to \mathrm{Ex}^\infty(X^{\vee r})$ and not on the codomain.

The next few pages consist of defining the operadic composition maps. We start by recalling some notation.

Observation 5.1.3 Recall from Chapter 1 that $\mathrm{Ex}^\infty(X)$ is defined the colimit of the following chain of acyclic cofibrations

$$X \xrightarrow{\sim} \mathrm{Ex}(X) \xrightarrow{\sim} \mathrm{Ex}^2(X) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathrm{Ex}^i(X) \xrightarrow{\sim} \cdots$$

Since cofibrations are injective in the Kan-Quillen model structure, this means that for all $x \in \mathrm{Ex}^\infty(X)$ there exists an $N > 0$ such that $x \in \mathrm{Ex}^n(X)$ for all $n > N$. Of course, we are implicitly identifying each $\mathrm{Ex}^n(X)$ with its image in $\mathrm{Ex}^\infty(X)$, where they form an exhaustive filtration.

Definition 5.1.4 Let X be a simplicial set with only finitely many non-degenerate simplices, and let f be an n-simplex of $\mathrm{CoEnd}(X)(r)$. In other words,

$$f \in \mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(X^{\vee r}))_n.$$

By the definition of simplicial mapping sets (see Definition 3.1.12), f is a simplicial function $X \times \triangle^m \to \mathrm{Ex}^\infty(X^{\vee r})$. Following Observation 5.1.3, we can associate an integer N_σ to every simplex $\sigma \in X \times \triangle^m$; this being the smallest N such that $f(\sigma) \in \mathrm{Ex}^N(X^{\vee r})$. We define N_f to be the integer $\max\{N_\sigma\}_{\sigma \in X \times \triangle^m}$.

Remark 5.1.5 The integer N_f is well-defined because $X \times \triangle^m$, the domain of f, has only finitely many non-degenerate simplices.

Remark 5.1.6 It is easy to scheck the following three properties of N_f.

- f factors through $\mathrm{Ex}^{N_f}(X^{\vee r})$.
- N_f is the smallest integer with this property.
- For all $N \geq N_f$, f factors through $\mathrm{Ex}^N(X^{\vee r})$.

Our definition of the coendomorphism operad will make heavy use of the adjunction between Ex and sd. For ease of reading, we shall introduce two pieces of helpful notation.

Notation 5.1.7 Let $f \in \mathrm{Hom}_{\mathrm{Set}_\triangle}(\mathrm{sd}^N(X \times \triangle^m), (X^{\vee r}))$ for $N > 0$. This is adjoint to $f^c \in \mathrm{Hom}_{\mathrm{Set}_\triangle}((X \times \triangle^m), \mathrm{Ex}^m(X^{\vee r}))$. Now f^c uniquely extends to an element of $\mathrm{Hom}_{\mathrm{Set}_\triangle}((X \times \triangle^m), \mathrm{Ex}^\infty(X^{\vee r}))$ which is the same thing as $\mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(X^{\vee r}))_n$. We shall denote this element as $\overline{f}$.

Notation 5.1.8 Let $f \in \mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(X^{\vee r}))_m$. Then it follows from Remark 5.1.5 that for all $N \geq N_f$, there is a unique element, which we shall denote (f, N), of $\mathrm{Hom}_{\mathrm{Set}_\triangle}(\mathrm{sd}^N(X \times \triangle^m), X^{\vee r})$, such that $\overline{(f, N)} = f$.

Having dispensed with the preliminaries we are now in a position to define the composition maps. Observe that the subdivision functor is a left adjoint and so preserves colimits. In particular, it commutes with wedge products.

Definition 5.1.9 Let $f \in \mathrm{CoEnd}(X)(r)_m$ and $f_i \in \mathrm{CoEnd}(X)(n_i)_m$ for $1 \leq i \leq r$. We define the composition map

$$\gamma : \mathrm{CoEnd}(X)(r) \times \mathrm{CoEnd}(X)(n_1) \times \cdots \mathrm{CoEnd}(X)(n_r) \to \mathrm{CoEnd}(X)(n_1 + \cdots + n_r)$$

to be $\overline{F}$ where F is the map

$$F : \mathrm{sd}^{N+N_f}(X \times \triangle^m) \xrightarrow{\mathrm{sd}^N(\delta_{\triangle^m})} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m))$$
$$\xrightarrow{\mathrm{sd}^N(\mathrm{id} \times \mathrm{sd}^{N_f}(\pi_2))} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(\triangle^m)) \xrightarrow{a} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)$$
$$\xrightarrow{(f,N_f)} \mathrm{sd}^N(X^{\vee r} \times \triangle^m)) \xrightarrow{b} \mathrm{sd}^N(X \times \triangle^m)^{\vee r} \xrightarrow{(f_1,N)\vee\cdots\vee(f_r,N)} X^{\vee n_1+\cdots n_r}$$

where

- N is the integer $\max(N_{f_1}, \ldots, N_{f_n})$.
- $\delta_{\mathrm{sd}^{N_f}(X\times\triangle^m)} : \mathrm{sd}^{N_f}(X \times \triangle^m) \to \mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m)$ is the diagonal map.
- $\pi_2 : X \times \triangle^m \to \triangle^m$ is the projection.
- $a : \mathrm{sd}^{N_f}(\triangle^m)) \to \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)$ is the map $\mathrm{sd}^N(\mathrm{id} \times \nu_{\triangle^m}^{(N_f)})$ where $\nu_{\triangle^m}^{(N_f)} := \nu_{\triangle^m} \circ \cdots \circ \nu_{\mathrm{sd}^{N_f-1}\triangle^m}$ and $\nu_Z : \mathrm{sd}\, Z \to Z$ is the last vertex map.
- b is an isomorphism, as $\times$ is distributive over the wedge product, and the wedge product commutes with subdivision.

We need to check that the definition above gives rise to well-defined operad. We phrase this result as a theorem.

Theorem 5.1.10 *Let X be a simplicial set with finitely many non-degenerate simplices. Then the composition maps of Definition 5.1.9 induce an operad structure on the* $\mathbb{S}$*-module* CoEnd(X)*.*

Before proving this theorem, we wish to make two useful remarks and introduce a final piece of notation.

Remark 5.1.11 Our first remark concerns the relationship between (f, N) and (f, M) for $M > N \geq N_f$. From the definition of Ex we see that, for all simplicial sets Z and Z', the simplicial morphism $\mathrm{Hom}_{\mathrm{Set}_\triangle}(\nu_Z, Z')$ is adjoint to $\mathrm{Hom}_{\mathrm{Set}_\triangle}(Z, \mu_{Z'})$, where both

$$\mu_Z : Z \to \mathrm{Ex}(Z).$$

$$\nu_Z : \mathrm{sd}\, Z \to Z.$$

are the maps induced by the last vertex map. Thus we have the relation

$$(f, N) \circ \nu_{\mathrm{sd}^N(X\times\triangle^m)} = (f, N + 1).$$

for all $N \geq N_f$ and its obvious extension by induction. A second useful well-known result about ν_Z that you should keep in the back of your mind is that the following diagram commutes

$$\begin{array}{ccc} \mathrm{sd}\, Z & \xrightarrow{\nu_Z} & Z \\ \Big\downarrow{\scriptstyle \mathrm{sd}\, f} & & \Big\downarrow{\scriptstyle f} \\ \mathrm{sd}\, Z' & \xrightarrow{\nu_{Z'}} & Z'. \end{array} \tag{5.1}$$

Notation 5.1.12 We define $\nu_Z^{(k)} := \nu_Z \circ \cdots \circ \nu_{\mathrm{sd}^{k-1} Z}$.

Remark 5.1.13 Another useful thing is to note that we can replace N_f in the definition of F with any integer $K \geq N_f$, and F will not change. To see why, call this new map $F(K)$, and then observe, with the help of Diagram 5.1, that $F(K) = F \circ \nu^{(K-N_f)}_{\mathrm{sd}^{N_f}(X \times \triangle^m)}$. By our previous remark

$$\overline{F \circ \nu^{(K-N_f)}_{\mathrm{sd}^{N_f}(X \times \triangle^m)}} = \overline{F}.$$

Similarly, if we replace N in the definition with a larger integer K', the function F in Definition 5.1.9 will become another function which we will call $F(K')$. It once again follows from Remark 5.1.11 and Diagram 5.1 that this function will be related to F by the identity

$$f(K') = F \circ \nu_Z^{(K'-N)},$$

and so we can also replace N with any larger integer in Definition 5.1.9 without changing the operad structure.

Proof (Theorem 5.1.10) We need to verify that this defines an operad, starting with the associativity axiom. So we wish to show that

$$\gamma(\gamma(f, f_1, \ldots, f_r), f_{1,1}, \ldots, f_{r,n_r}) = \gamma(f, \gamma(f_1, f_{1,1}, \ldots f_{1,n_1}), \ldots, \\ \gamma(f_1, f_{r,1}, \ldots f_{r,n_r}))$$

for all $f \in \mathrm{CoEnd}(X)(r)_m$, $f_i \in \mathrm{CoEnd}(X)(n_i)_m$ and $f_{i,j} \in \mathrm{CoEnd}(X)(n_{i,j})_m$. Expanding the left hand side of this we obtain

$$(\bigvee_{i=1}^{r} \bigvee_{j=1}^{r_i} (f_{ij}, M)) \circ (\bigvee_{k=1}^{r} \mathrm{sd}^M((f_k, M') \times \nu_{\triangle^m}^{(M')} \circ \mathrm{sd}^{M'}(\pi_2))) \tag{5.2}$$

$$\circ\, \mathrm{sd}^{M+M'}((f, N_f) \times (\nu_{\triangle^m}^{(N_f)} \circ \mathrm{sd}^{N_f}(\pi_2))) \tag{5.3}$$

where $M = \max\{N_{f_{ij}}\}_{1 \geq i \geq r, 1 \leq j \leq r_i}$ and $M' = \max\{N_{f_i}\}_{1 \leq i \leq r}$. Now let $M_i = \max\{M_{ij}\}_{0 \leq j \leq r_i}$ and recall that

$$(f, M) = (f, M_i) \circ \nu^{(M-M_i)}_{\mathrm{sd}^M(X \times \triangle^m)}$$

We may deduce from this that Expression (5.3) can be written

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij},M_i)\circ \nu^{(M-M_i)}_{\mathrm{sd}^{M_i}(X\times\triangle^m)})\circ(\bigvee_{k=1}^{r}\mathrm{sd}^{M}((f_k,M')\times\nu^{(M')}_{\triangle^m}\circ\mathrm{sd}^{M'}(\pi_2)))$$
$$\circ\,\mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

This can be written

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij},M_i))\circ(\bigvee_{k=1}^{r}\nu^{(M-M_k)}_{\mathrm{sd}^{M}(X^{\vee r_k}\times\triangle^m)}\circ\mathrm{sd}^{M}((f_k,M')\times\nu^{(M')}_{\triangle^m}\circ\mathrm{sd}^{M'}(\pi_2)))$$
$$\circ\,\mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

Using the commutativity of Diagram 5.1 we see that this is equal to

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij},M_i))\circ(\bigvee_{k=1}^{r}\mathrm{sd}^{M_k}((f_k,M')\times\nu^{(M')}_{\triangle^m}\circ\mathrm{sd}^{M'}(\pi_2))\circ\nu^{M-M_k}_{\mathrm{sd}^{M'+M_k}(X\times\triangle^m)})$$
$$\circ\,\mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

Once again using Diagram 5.1, we can rewrite this as

$$(\bigvee_{i=1}^{r}(\bigvee_{j=1}^{r_i}(f_{ij},M_i))\circ\mathrm{sd}^{M_i}((f_i,M_{f_i})\times(\nu^{(M_{f_i})}_{\triangle^m}\circ\mathrm{sd}^{M_{f_i}}(\pi_2)))\circ\nu^{M+M'-M_i-M_{f_i}}_{\mathrm{sd}^{M'+M_i}(X\times\triangle^m)})$$
$$\circ\,\mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

The above expression is equal to

$$\bigvee_{i=1}^{r}(\overline{\gamma(f_i,f_{i1},\dots,f_{ir_i})},M+M')\circ\mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\mathrm{sd}^{M+M'}(\triangle^m)}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

By our argument on the last page, this is equal to

$$\gamma(f,\gamma(f_1,f_{1,1},\dots f_{1,n_1})\dots,\gamma(f_1,f_{r,1},\dots f_{r,n_r}))$$

as desired.

The identity element of the operad is

$$\mu_X : X\to \mathrm{Ex}^{\infty}(X).$$

Verifying the equivariance axioms is straightforward, it is almost exactly the same as verifying them for topological coendomorphism operad. Therefore we have defined an operad. □

It remains only to define simplicial coalgebras, which proceeds exactly as one would expect.

Definition 5.1.14 Let $\mathscr{P}$ be an operad in simplicial sets. We shall say that a finite simplicial set X is a *$\mathscr{P}$-coalgebra* if there exists an operadic morphism $\Phi : \mathscr{P} \to \mathrm{CoEnd}(X)$.

Lastly we define E_n-algebras in $\mathrm{Set}_\triangle$.

Definition 5.1.15 In simplicial sets, an *E_n-(co)algebra* is a homotopy (co)algebra over the Barratt-Eccles E_n-operad.

5.2 Simplicial suspensions are E_n-coalgebras

In this section, in direct analogy with Moreno-Fernández and Wierstra's result in topological spaces, we aim to show that simplicial suspensions are homotopy coalgebras over the Barratt-Eccles operad. The strategy of this proof is as follows. First we transfer the little n-discs operad $\mathbb{D}_n$, the topological coendomorphism operad and the operad morphism between them Φ into the category of simplicial sets using the simplicial chains functor $S_\bullet$. We then use the homotopy transfer principle to lift this to a morphism from a cofibrant replacement of $\mathbb{D}_n$ to the simplicial coendomorphism operad.

The precise statement of the simplicial version of Theorem 2.4.10 is as follows.

Theorem 5.2.1 *Let $n \in \mathbb{N}$ and $\Sigma^n X$ be the n–fold suspension of a finite simplicial set X. Then $\Sigma^n X$ has the structure of an E_n-coalgebra.*

Our proof of this theorem requires that the Cartesian product commutes with the geometric realization functor. This is actually not true in general. Therefore, we shall need to restrict from the category of all topological spaces to the category of compactly generated Hausdorff spaces.

Definition 5.2.2 Let $(\mathbf{Top}', \times_{Ke})$ be the full subcategory of Top whose objects are the compactly generated Hausdorff spaces. This is equipped with the *Kelley product* defined by

$$X \times_{Ke} Y = (X \times Y)_c$$

where $(X \times Y)_c$ is the set $X \times Y$ equipped with the following topology; a subset $A \subseteq (X \times Y)_c$ is closed if and only if for all compact subsets K of the topological space $X \times Y$, the set $A \cap K$ is closed in the topological space $X \times Y$.

Remark 5.2.3 The Kelley product is the same as the ordinary product if at least one of the factors is locally compact. Carefully examining Example

2.4.9 and the proof of Theorem 2.4.10, we see that there is therefore no reason why we cannot take the category we working over to be $\mathbf{Top}'$, with all other definitions remaining unchanged.

Remark 5.2.4 The key reason why we use the Kelly product here is that, if X and Y are simplicial sets,

$$|X| \times_{Ke} |Y| = |X \times Y|$$

something that is not true for the ordinary Cartesian product in topological spaces.

We also wish to be able to transfer operads from topological space to simplicial sets. This is made possible by the following definition.

Definition 5.2.5 Let $\mathscr{P}$ be an operad in Top or in $\mathbf{Top}'$. We define an operad $S_\bullet \mathscr{P}$ over $\mathrm{Set}_\triangle$ with arity n component

$$(S_\bullet \mathcal{P})(n) := S_\bullet(\mathcal{P}(n))$$

where $S_\bullet$ is the singular chains functor. The action of $\sigma \in \mathbb{S}_n$ on $S_\bullet \mathcal{P}(n)$ is given by $S_\bullet \mathcal{P}(n) * \sigma := S_\bullet(\mathcal{P}(n) * \sigma)$. The operadic composition map is $\gamma_{S_\bullet \mathcal{P}} := S_\bullet(\gamma_{\mathcal{P}})$ and we take the unit to be the simplex $[\triangle^0 \to 1_{\mathbf{Top}'}] \in S_\bullet \mathcal{P}(1)$.

Remark 5.2.6 The operad composition map in the definition above is well-defined because $S_\bullet$ is right adjoint to the geometric realization. This means that it preserves limits, and in particular, products.

We can actually define $S_\bullet(\mathrm{CoEnd}_{\mathbf{Top}'}(|X|))$ to be an alternative coendomorphism operad. The following theorem gives us a precise description of it.

Lemma 5.2.7 *Let X be a simplicial set with only finitely many nondegenerate simplices. The operad $S_\bullet(\mathrm{CoEnd}_{\mathbf{Top}'}(|X|))$ is isomorphic to the simplicial operad $Q(X)$ with arity r component equal to*

$$Q(X)(r) := \mathrm{Map}_{\mathrm{Set}_\triangle}(X, S_\bullet |X^{\vee r}|).$$

Let $f \in Q(X)(r)_m$ and let $f_i \in Q(X)(n_i)_m$ for $1 \le i \le r$. The operadic composition map

$$\gamma : Q(X)(r) \times Q(X)(n_1) \times \cdots \times Q(X)(n_r) \to Q(X)(n_1 + \cdots + n_r)$$

is given by the adjoint under the Top-$\mathrm{Set}_\triangle$ *adjunction of $F : |X \times \triangle^m| \to |X^{\vee n_1 + \cdots + n_r}|$, where F is defined by*

$$\begin{aligned} &|X \times \triangle^m| \xrightarrow{|\mathrm{id} \times \delta_{\triangle^m}|} |X \times \triangle^m \times \triangle^m| \xrightarrow{a} |X \times \triangle^m| \times_{Ke} |\triangle^m| \xrightarrow{|f| \times_{Ke} \mathrm{id}} \\ &\quad |S_\bullet |X^{\vee r}|| \times_{Ke} |\triangle^m| \xrightarrow{\varepsilon_{X^{\vee r}} \times_{Ke} \mathrm{id}} |X^{\vee r}| \times_{Ke} |\triangle^m| \xrightarrow{b} |X \times \triangle^m|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |f_i|} \\ &\qquad \bigvee_{i=1}^r |S_\bullet |X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r \varepsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}| \xrightarrow{c} |X^{\vee n_1 + \cdots + n_r}| \end{aligned}$$

where

- $\delta_{\triangle^m} : \triangle^m \to \triangle^m \times \triangle^m$ *is the diagonal map.*
- *for* Y *a topological space, the map* $\varepsilon_Y : |S_\bullet(Y)| \to Y$ *is the counit of the adjunction between topological spaces and simplicial sets.*
- $a : |X \times \triangle^m \times \triangle^m| \to |X \times \triangle^m| \times_{Ke} |\triangle^m|$ *is an isomorphism, as* $\times$ *commutes with geometric realisation.*
- $b : |X^{\vee r}| \times_{Ke} |\triangle^m| \to |X \times \triangle^m|^{\vee r}$ *is an isomorphism, as both* $\times$ *and the wedge product commute with geometric realisation.*
- $c : \bigvee_{i=1}^r |X^{\vee n_i}| \to |X^{\vee n_1+\cdots+n_r}|$ *is an isomorphism, as the wedge product commutes with geometric realisation.*

For each $\sigma \in \mathbb{S}_r$*, there is a map* $\sigma^* : X^\vee \to X^\vee$ *given by permuting the terms of the wedge sum by* σ*. The symmetric structure on* $Q(X)(r)$ *is defined by post-composition with the morphism* $S_\bullet|\sigma^*|$*.*

Proof We can write

$$S_\bullet(\mathrm{CoEnd}_{\mathbf{Top}'}(|X|))(r) = S_\bullet \mathrm{Map}_{\mathbf{Top}'}(|X|, |X^{\vee r}|) \cong \mathrm{Map}_{\mathrm{Set}_\triangle}(X, S_\bullet|X^{\vee k}|).$$

because, for all $K \in \mathrm{Set}_\triangle$ and $Y \in \mathbf{Top}'$, we have

$$\mathrm{Hom}_{\mathbf{Top}'}(|\triangle^m|, \mathrm{Map}_{\mathbf{Top}'}(|K|, Y)) \cong \mathrm{Hom}_{\mathbf{Top}'}(|\triangle^m| \times_{Ke} |K|, Y)$$

by tensor-hom adjunction. Here it is critical to distinguish between the simplicial mapping space and the hom-set. We then have

$$\mathrm{Hom}_{\mathbf{Top}'}(|\triangle^m| \times_{Ke} |K|, Y) \cong \mathrm{Hom}_{\mathbf{Top}'}(|\triangle^m \times K|, Y)$$

by the identity $|X| \times_{Ke} |Y| \cong |X \times Y|$ and finally we have

$$\mathrm{Hom}_{\mathbf{Top}'}(|\triangle^m \times K|, Y) \cong \mathrm{Hom}_{\mathrm{Set}_\triangle}(\triangle^m \times K, S_\bullet Y)$$

by adjunction.

Secondly, it remains to check that operad morphisms are as described in the statement of the lemma. We can describe the induced operad structure on $\mathrm{Hom}_{\mathbf{Top}'}(|\triangle^m \times X|, |X^{\vee r|})$ quite easily. For $f \in \mathrm{Hom}_{\mathbf{Top}'}(|\triangle^m \times X|, |X^{\vee r|})$ and $f_i \in \mathrm{Hom}_{\mathbf{Top}'}(|\triangle^m \times X|, |X^{\vee n_i|})$ the composite $\gamma(f, f_1, \ldots f_n)$ is the function

$$F : |X \times \triangle^m| \xrightarrow{|\mathrm{id} \times \delta_{\triangle^m}|} |X \times \triangle^m \times \triangle^m| \xrightarrow{a} |X \times \triangle^m| \times_{Ke} |\triangle^m| \xrightarrow{f \times_{Ke} \mathrm{id}}$$
$$|X^{\vee r}| \times_{Ke} |\triangle^m| \xrightarrow{b} |X \times \triangle^m|^{\vee r} \xrightarrow{\bigvee_{i=1}^r f_i} \bigvee_{i=1}^r |X^{\vee n_i}| \xrightarrow{c} |X^{\vee n_1+\cdots+n_r}|$$

The isomorphism

$$G : \mathrm{Hom}_{\mathrm{Set}_\triangle}(\triangle^m \times X, S_\bullet|X^{\vee r}|) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Top}'}(|\triangle^m \times X|, |X^{\vee r}|)$$

can be written by

$$f \mapsto \varepsilon_{X^{\vee r}} \circ |f|.$$

Therefore the composition map is exactly as described. □

Remark 5.2.8 It is important to remark that this operad will usually have a different homotopy type to the naïve simplicial coendomorphism operad mentioned at the start of this chapter.

Despite Remark 5.2.8, the simplicial coendomorphism operad and the operad $S_\bullet(\mathrm{CoEnd}_{\mathbf{Top}'}(|X|))$ **will** be equivalent.

Lemma 5.2.9 *Let X be a finite simplicial set. Then the simplicial coendomorphism operad and the operad $S_\bullet(\mathrm{CoEnd}_{\mathbf{Top}'}(|X|))$ are weakly equivalent.*

We shall prove this by constructing a zig-zig involving a third operad, which we define will first.

Definition 5.2.10 Let X be a finite simplicial set. Then the *mixed coendomorphism operad* $R(X)$ has arity r component

$$R(X)(r) = \mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(S_\bullet|X^{\vee r}|)).$$

For each $\sigma \in \mathbb{S}_r$, there is a map $\sigma^* : X^\vee \to X^\vee$ given by permuting the terms of the wedge sum by σ. The symmetric structure on $R(X)(r)$ is defined by post-composition with the morphism $\mathrm{Ex}^\infty(S_\bullet|\sigma^*|)$. We shall define the operadic composition map using both the sd–Ex and the simplicial chains-geometric realization adjunctions consecutively. Let $f \in Q(X)(r)_m$ and $f_i \in Q(X)(n_i)_m$ for $1 \leq i \leq r$, then the operadic composition map

$$\gamma : R(X)(r) \times R(X)(n_1) \times \cdots \times R(X)(n_r) \to R(X)(n_1 + \cdots + n_r)$$

is defined to be $\overline{F}$ which is adjoint, under the sd-Ex adjunction, of the morphism, $F : \mathrm{sd}^{N_f}(X \times \triangle^m) \to S_\bullet|X^{\vee n_1 + \cdots + n_r}|$. F is itself an adjoint, this time under the geometric realization –simplicial chains adjunction, of a morphism $G : |\,\mathrm{sd}^{N+N_f}(X \times \triangle^m)| \to |X^{\vee n_1 + \cdots + n_r}|$ which we define to be the composite

$$\begin{aligned}
&|\,\mathrm{sd}^{N+N_f}(X \times \triangle^m)| \xrightarrow{|\,\mathrm{sd}^N(\delta_{\triangle^m})|} |\,\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m))| \\
&\xrightarrow{|\,\mathrm{sd}^N(\mathrm{id} \times \mathrm{sd}^{N_f}(\pi_2))|} |\,\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(\triangle^m))| \xrightarrow{a} |\,\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)| \\
&\xrightarrow{\mathrm{sd}^N((f, N_f) \times \mathrm{id})} |\,\mathrm{sd}^N(S_\bullet|X^{\vee r}| \times \triangle^m)| \xrightarrow{b} |S_\bullet|X^{\vee r}| \times \triangle^m| \xrightarrow{c} |X^{\vee r} \times \triangle^m| \\
&\xrightarrow{d} |\,\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \xrightarrow{e} |\,\mathrm{sd}^N(X^{\vee r} \times \triangle^m)|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |f_i|} \bigvee_{i=1}^r |S_\bullet|X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r \varepsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}|
\end{aligned}$$

where

- N is the integer $\max(N_{f_1}, \ldots, N_{f_n})$.
- and for Y a topological space, the map $\varepsilon_Y : |S_\bullet(X^{\vee r})| \to Y$ is the counit of the adjunction between topological spaces and simplicial sets.
- $\delta_{\mathrm{sd}^{N_f}(X\times\triangle^m)} : \mathrm{sd}^{N_f}(X \times \triangle^m) \to \mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m)$ is the diagonal map.
- $\pi_2 : X \times \triangle^m \to \triangle^m$ is the projection.
- $a : |\,\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(\triangle^m))| \to |\,\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)|$ is the map $|\,\mathrm{sd}^N(\mathrm{id} \times \nu_{\triangle^m} \circ \cdots \circ \nu_{\mathrm{sd}^{N_f-1}\triangle^m})|$.
- $b : |\,\mathrm{sd}^N(S_\bullet|X^{\vee r}| \times \triangle^m)| \to |S_\bullet|X^{\vee r}| \times \triangle^m|$ is a homeomorphism, by Lemma 3.4.2, which states that there is a homeomorphism $h_Z : |\,\mathrm{sd}(Z)| \to |Z|$ for every simplicial set Z (although this homeomorphism is not necessarily natural for simplicial morphisms $Z \to Z'$).
- $c : |S_\bullet|X^{\vee r}| \times \triangle^m| \xrightarrow{c} |X^{\vee r} \times \triangle^m|$ is the composite

$$|S_\bullet|X^{\vee r}| \times \triangle^m| \xrightarrow{p} |S_\bullet|X^{\vee r}|| \times_{Ke} |\triangle^m| \xrightarrow{|\varepsilon_{X^{\vee r}}| \times_{Ke} \mathrm{id}} |X^{\vee r}| \times_{Ke} |\triangle^m| \xrightarrow{q} |X^{\vee r} \times \triangle^m|$$

  where p and q are isomorphisms as the Kelley product commutes with geometric realisation.
- $d : |X^{\vee r} \times \triangle^m| \to |\,\mathrm{sd}^N(X^{\vee r} \times \triangle^m)|$ is the homeomorphism that exists by Lemma 3.4.2.
- $e : |\,\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \to |\,\mathrm{sd}^N(X^{\vee r} \times \triangle^m)|^{\vee r}$ is a homeomorphism because wedge product commutes with geometric realization.
- $f : \bigvee_{i=1}^r |X^{\vee n_i}| \to |X^{\vee n_1+\cdots+n_r}|$ is a homeomorphism, as the wedge product commutes with geometric realisation.

We now start the proof of Lemma 5.2.9.

Proof (Lemma 5.2.9) Since, by Lemma 5.2.7, the operad $S_\bullet(\mathrm{CoEnd}_{\mathbf{Top}'}(|X|))$ is isomorphic to $Q(X)(r)$, it suffices to construct a zig-zag of weak equivalences

$$\mathrm{CoEnd}(X) \xrightarrow{p} R(X) \xleftarrow{q} Q(X).$$

We define $p(r)$ to be the morphism

$$\mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(v_{X^{\vee r}})) : \mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(X^{\vee r})) \to \mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(S_\bullet|X^{\vee r}|))$$

where $v_{X^{\vee r}} : X^{\vee r} \to S_\bullet|X^{\vee r}|$ is the unit of the singular chains – geometric realization adjunction. Observe that $\mathrm{Ex}^\infty(v_{X^{\vee r}}) : \mathrm{Ex}^\infty(X^{\vee r}) \to \mathrm{Ex}^\infty(S_\bullet|X^{\vee r}|)$

is a weak equivalence between fibrant simplicial sets. Hence it is a homotopy equivalence, and the functor $\mathrm{Map}_{\mathrm{Set}_\triangle}(X,-)$ preserves homotopy equivalences. Hence p is a weak equivalence.

It remains to check that it induces a morphism of operads. We check this directly. Note first that $\mathrm{Map}_{\mathrm{Set}_\triangle}(X, \mathrm{Ex}^\infty(v_{X^{\vee r}}))(f) = \mathrm{Ex}^\infty(v_{X^{\vee r}}) \circ f$. Then observe that $N_{\mathrm{Ex}^\infty(v_{X^{\vee r}})\circ f} = N_f$ and $\max(N_{v_{X^{\vee r}})\circ f_1}, \ldots, N_{v_{X^{\vee r}})\circ f_n}) = \max(N_{f_1}, \ldots, N_{f_n})$. Then observe that the morphism

$$|\,\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)| \xrightarrow{\mathrm{sd}^N((\mathrm{Ex}^\infty(v_{X^{\vee r}})\circ f, N_f)\times \mathrm{id})} |\,\mathrm{sd}^N(S_\bullet|X^{\vee r}| \times \triangle^m)|$$

factors as

$$\begin{aligned} |\,\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)| &\xrightarrow{\mathrm{sd}^N((f, N_f)\times \mathrm{id})} |\,\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \\ &\xrightarrow{\mathrm{sd}^N(v_{X^{\vee r}}\times \mathrm{id})} |\,\mathrm{sd}^N(S_\bullet|X^{\vee r}| \times \triangle^m)| \end{aligned}$$

Moreover, having first observed that the following diagram is commutative

$$\begin{array}{ccc} |\,\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| & \xrightarrow{|\,\mathrm{sd}^N(v_{X^{\vee r}}\times \mathrm{id})|} & |\,\mathrm{sd}^N(S_\bullet|X^{\vee r}| \times \triangle^m)| \\ \downarrow h_{(X^{\vee r}\times \triangle^m)} & & \downarrow h_{S_\bullet|X^{\vee r}|\times\triangle^m} \\ |(X^{\vee r} \times \triangle^m)| & \xrightarrow{|(v_{X^{\vee r}}\times \mathrm{id})|} & |(S_\bullet|X^{\vee r}| \times \triangle^m)|, \end{array}$$

where $h_Z : |\,\mathrm{sd}\, Z| \to |Z|$ is the map that exists by Lemma 3.4.2, we see that the composite

$$|\,\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \xrightarrow{|\,\mathrm{sd}^N(v_{X^{\vee r}}\times \mathrm{id})} |\,\mathrm{sd}^N(S_\bullet|X^{\vee r}| \times \triangle^m)| \xrightarrow{b} |S_\bullet|X^{\vee r}| \times \triangle^m|$$
$$\xrightarrow{c} |S_\bullet|X^{\vee r}|| \times_{Ke} |\triangle^m| \xrightarrow{|\varepsilon_{X^{\vee r}}| \times_{Ke} \mathrm{id}} |X^{\vee r}| \times_{Ke} |\triangle^m| \xrightarrow{d} |X^{\vee r} \times \triangle^m| \xrightarrow{e} |\,\mathrm{sd}^N(X \times \triangle^m)|^{\vee r}$$

is an isomorphism by the *triangle identities* for the $S_\bullet$–$|-|$ adjunction. Explicitly, the (left) triangle identity for an adjunction $L \dashv R$ with unit $\eta : id_X \to R \circ L$ and counit $\varepsilon : L \circ R \to id_Y$ states that the natural transformation of functors defined as the composite

$$L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L$$

is the identity transformation. Upon further observing that, for the same reason, the composite

$$|\,\mathrm{sd}^N(X \times \triangle^m)|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |\,\mathrm{Ex}^\infty(v_{X^{\vee r}})\circ f_i|} \bigvee_{i=1}^r |S_\bullet|X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r \varepsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}|$$

is exactly the map

$$|\operatorname{sd}^N(X \times \triangle^m)|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |f_i|} \bigvee_{i=1}^{r} |X^{\vee n_i}|,$$

it becomes obvious that γ commutes with p, and so p is a weak equivalence of operads.

Similarly, we define $q(r)$ to be the morphism

$$\operatorname{Map}_{\operatorname{Set}_\triangle}(X, \mu_{S_\bullet|X^{\vee r}|}) : \operatorname{Map}_{\operatorname{Set}_\triangle}(X, S_\bullet|X^{\vee r}|) \to \operatorname{Map}_{\operatorname{Set}_\triangle}(X, \operatorname{Ex}^\infty(S_\bullet|X^{\vee r}|)).$$

This is a weak equivalence of simplicial sets for exactly the same reasons that $p(r)$ is. Observe that $N_{q(r)(f)} = 0$ for all $f \in Q(X)(r)$. It follows from the form of the operad maps that the morphism q identifies $Q(X)$ with a suboperad of $R(X)(r)$. In particular, q is a morphism of operads, and so a weak equivalence of operads. □

Finally, we can prove the main result of this section.

Proof (Theorem 5.2.1) Let $\Sigma^n X$ be the n–fold suspension of a simplicial set X. As $|\Sigma X|$ is CW-complex, it is in **Top**$'$. Suspensions are a particular kind of finite limits, and the geometric realization functor commutes with finite limits, so suspensions commute with geometric realization (alternatively, see [10]) and thus that $|\Sigma^n X|$ is a coalgebra over the little n–discs operad in **Top**$'$. This coalgebra structure is an an operadic morphism $\Phi : \mathbb{D}_n \to \operatorname{CoEnd}_{\mathbf{Top}'}(|\Sigma^n X|)$. As discussed above, we can use $S_\bullet$ to transfer these operads and this algebra structure to the category of simplicial sets, producing the following morphism of operads

$$S_\bullet(\Phi) : S_\bullet(\mathbb{D}_n) \to S_\bullet(\operatorname{CoEnd}_{\mathbf{Top}'}(|\Sigma^n X|))$$

Lemma 5.2.9 tells us that there is a weak equivalence between $\operatorname{CoEnd}(X)$ and $S_\bullet(\operatorname{CoEnd}_{\mathbf{Top}'}(|\Sigma^n X|))$. Observe that in each arity $\operatorname{CoEnd}(X)(n)$ is a mapping space where the target is a Kan complex, hence Kan itself and a fibrant operad in the operadic model structure. By its construction, in each arity $S_\bullet \operatorname{CoEnd}_{\mathbf{Top}'}(|\Sigma^n X|)$ is a singular complex and thus as an operad it is also fibrant.

Since we have a weak equivalence between fibrant operads, over the cofibrant replacement $(S_\bullet\mathbb{D}_n)_\infty$ of $S_\bullet\mathbb{D}_n$ we have an induced bijection between the homotopy classes of morphisms of operads

$$[(S_\bullet\mathbb{D}_n)_\infty, \operatorname{CoEnd}_{\operatorname{Set}_\triangle}(\Sigma^n X)] \cong [(S_\bullet\mathbb{D}_n)_\infty, S_\bullet \operatorname{CoEnd}_{\mathbf{Top}'}(|\Sigma^n X|)].$$

So we can choose a morphism $\varphi : (S_\bullet\mathbb{D}_n)_\infty \to \operatorname{CoEnd}_{\operatorname{Set}_\triangle}(\Sigma^n X)$, such that φ is homotopy equivalent to $S_\bullet\Phi$.

Finally to prove that n-fold suspensions are E_n-algebras it suffices to note that all topological operads are fibrant and so the weak equivalence between the little n-discs operad and the geometric realization of the Barratt-Eccles E_n-operad remains one when taking the $S_\bullet$ functor. The Barratt-Eccles E_n-operad $\Gamma^{(n)}$ is weakly equivalent to $S_\bullet|\Gamma^{(n)}|$, and in particular, $(S_\bullet \mathbb{D}_n)_\infty$ can be taken to be the Boardman-Vogt resolution of $\Gamma^{(n)}$; the operad $W(\triangle^1, \Gamma^{(n)})$ that we computed earlier. $\square$

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