A p-adic de Rham complex

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Abstract

This is the second in a sequence of three articles exploring the relationship between commutative algebras and E_∞-algebras in characteristic p and mixed characteristic. Given a topological space X, we construct, in a manner analogous to Sullivan's A_{PL}-functor, a strictly commutative algebra over $\widehat{\mathbb{Z}}_p$ which we call the de Rham forms on X. We show this complex computes the singular cohomology ring of X. We prove that it is quasi-isomorphic as an E_∞-algebra to the Berthelot-Ogus-Deligne *décalage* of the singular cochains complex with respect to the p-adic filtration. We show that one can extract concrete invariants from our model, including Massey products which live in the torsion part of the cohomology. We show that if X is formal then, except at possibly finitely many primes, the p-adic de Rham forms on X are also formal. We conclude by showing that the p-adic de Rham forms provide, in a certain sense, the "best functorial strictly commutative approximation" to the singular cochains complex.

1 Introduction

Since its introduction by Quillen [20] and Sullivan [21], rational homotopy theory has probably become the single most successful subfield of algebraic topology. One of the main observations of [21], which was completely fleshed out by [6], was that it was possible to completely capture the rational homotopy theory of spaces via a strictly commutative model $A_{PL}(X)$, which behaves roughly like the de Rham cochains. This reduces the study of rational topological spaces to that of commutative dg-algebras. This has led to some spectacular practical advances; for example, the rational homotopy groups of spheres and many other spaces are now completely understood.

In a tour de force, Mandell [17] showed that it was possible to go one step further, and that the study of all nilpotent, finite type spaces *integrally* can be reduced to studying E_∞-algebras. In terms of computation, less mileage seems to have been got from this than rationally; largely because E_∞-algebras are usually very complicated objects, generated by infinitely many n-ary operations, and which are not naturally amenable to being studied computationally. We are unaware of any implementations of even simple procedures such as Groebner bases for general E_∞-algebras. In contrast, the strictly commutative algebras appearing in rational homotopy theory are, almost uniquely, suited to being studied via computer algebraic approaches such as using GAP or Sage due to the fact they are generated by a single binary operation displaying the simplest possible behaviour. Most of these techniques are not available even one step up, when working with cup-1-algebras - algebras that are commutative up to strictly commutative homotopy [11, Definition 4.18].

The goal of this article is therefore to provide strictly commutative models for spaces over the p-adic numbers $\widehat{\mathbb{Z}}_p$. The central problem is that it is not possible to capture all of the information about the homotopy type of the spaces this way. This because the Steenrod operations act as obstructions to strict commutivity. In particular, we have that the zeroth Steenrod power operation P^0 never vanishes on E_∞-algebras with the homotopy type of spaces. Therefore, we can only hope to study approximations that carry *some* of this information. There are multiple possible approaches. Mandell [18] has suggested for n-connected spaces X at most primes, it may be possible to (non-functorially) truncate the E_∞-structure on $C^*\left(X, \widehat{\mathbb{Z}}_p\right)$ to an E_n-structure and find a strictly commutative model for this truncation. While we think this is a interesting point of view

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and worthy of further study, in this paper we have opted for a more functorial approach. We further explain which well-known invariants may be extracted from it.

In this paper, we study a generalisation of Sullivan's approach to homotopy theory. Recall that this involves defining a *cochain algebra*, that is a functor

$$A_{PL} : \triangle \to \mathsf{CDGA}$$

which extends to

$$A_{PL} : \mathsf{sSet} \to \mathsf{CDGA}$$

by the universal property of simplicial sets. We shall recall this in more detail later, but for now it suffices to recall that

$$A_{\mathrm{PL}}\left(\Delta^n\right) = \frac{\mathbb{Q}\left(t_0, ..., t_n, d\,t_0, ..., d\,t_n\right)}{\left(\sum t_i - 1, \sum d\,t_i\right)}$$

The problem with doing this in positive characteristic is that Sym is not a homotopy invariant functor. In 1979, Cartan [8] generalised the work of Sullivan [21] to a slightly more general framework. In particular, Example 4 from that paper uses divided power algebras

$$\mathrm{Gr}\left(\Delta^n\right) = \frac{\mathbb{Z}\langle s\rangle\langle t_0, ..., t_n, d\,t_0, ..., d\,t_n\rangle}{\left(\sum t_i - s, \sum d\,t_i\right)}$$

where $\langle-\rangle$ denotes the free divided power algebra. Cartan computes the cohomology of the extension to sSet and proves that a subring of the cohomology is isomorphic to the singular cohomology ring of X.

We, initially independently, had the same idea of modifying Sullivan's construction using divided power algebras. However, instead of working with $\mathbb{Z}\langle s\rangle$, we found it more convenient to localise at a fixed prime p and work over $\widehat{\mathbb{Z}_p}$, with p itself playing the role of s. This way, we are able to extract the singular cohomology ring of $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$ itself from the construction, which we call *the $\widehat{\mathbb{Z}_p}$-de Rham forms on X*.

Theorem 1.1. *Let X be a simplicial set. The cohomology ring of the p-adic de Rham complex $\Omega^*(X)$ is isomorphic to the singular cohomology of X. In other words, one has a ring isomorphism*

$$H^*\left(\Omega^*(X)\right) \cong H^*\left(X, \widehat{\mathbb{Z}_p}\right).$$

After computing the cohomology ring, from a modern perspective, the natural next step is interpret the higher information contained the $\widehat{\mathbb{Z}_p}$-de Rham forms. To that end, we show (Theorem 3.17) that our construction, as an E_∞-algebra, is equivalent to the following subalgebra of the singular cochains. In this sense, our work is the logical continuation of that by Cartan and sheds new light on many of the constructions of [8].

Definition 1.2. Let X be a simplicial set. We define the *p-shifted singular cochain algebra* $\mathscr{D}^*\left(X, \widehat{\mathbb{Z}_p}\right)$ to be the following subalgebra of the singular cochains $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$.

$$\mathscr{D}^n(X) = \left\langle p^i\sigma : \text{ for } \sigma \in C^n\left(X, \widehat{\mathbb{Z}_p}\right) \text{ and } \begin{cases} i = n & \text{if } d\sigma = 0. \\ i = n+1 & \text{otherwise.} \end{cases} \right\rangle$$

The differential and the E_∞-structure are that induced by those on $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$.

This also reveals an unexpected connection with the theory of crystalline cohomology for schemes. The same object as above can be viewed as $\eta_p\left(C^*\left(X, \widehat{\mathbb{Z}_p}\right)\right)$, where η is the Berthelot-Ogus-Deligne [2, 9] *décalage* functor, which is defined as the connective cover with respect to the Beilinson t-structure on filtered complexes. In our case we we are working in complexes over $\widehat{\mathbb{Z}_p}$ with the p-adic filtration. In Cartan's case, he was working in complexes over $\mathbb{Z}\langle s\rangle$ with the filtration generated by the ideal (s). In particular, this ties in with the work of Bhatt-Lurie-Mathew [3, Thereom 7.4.7, Example 7.6.7], which states that, in the ∞-categorical context, the fixed points of the left derived functor $L\eta_p$ of η_p acting on the derived category of p-complete dg-$\widehat{\mathbb{Z}_p}$-modules

is equivalent to a 1-category. The de Rham forms appearing in our and Cartan's work can therefore be seen supplying a convenient strictly commutative model for this rectification when working with spaces.

Theorem 3.17 also has some immediate applications. It means that the $\widehat{\mathbb{Z}_p}$-de Rham forms can be used to compute Massey products up to a factor, including in the torsion part of the cohomology, which has proven useful, in, for example, [14] for specific classes of spaces. We conclude with a result on formality which was inspired by a conjecture of Mandell's [18].

Theorem 1.3. *Let X be a finite simplicial set such that $A_{PL}(X)$ is formal over $\mathbb{Q}$. For all but finitely many primes, $\Omega^*(X)$ is formal over $\widehat{\mathbb{Z}_p}$ as a dg-commutative dg-algebra.*

Finally, we show that the $\widehat{\mathbb{Z}_p}$-de Rham forms are the *best functorial strictly commutative approximation* to the singular chains. We do this by exhibiting examples of spaces X such that $C^*(X, \widehat{\mathbb{Z}_p})$ cannot be rectified as an associative algebra. The obstructions in question are incompatible Massey products in $C^*(X, \mathbb{F}_p)$, which is precisely the information lost by the $\widehat{\mathbb{Z}_p}$-de Rham forms.

Structure of the article

This paper has the following structure. First we recall some preliminaries on rational homotopy theory, divided power algebras and E_∞-algebras. Then in part 3, we define the de Rham forms, compute ther cohomology and relate them to a subalgebra of the singular cochains complex. Finally, in part 4, we examine the homotopy invariants that can be extracted from the p-adic de Rham forms and prove a formality theorem. In part 4, we construct an example of a space X with non-vanishing higher Steenrod operations on its E_1-algebra structure.

Acknowledgements

Notation and conventions

In this paper, we work on the category of unbounded cochain complexes over some base field or ring with cohomological convention. That is, the differential $d : A^* \to A^{*+1}$ of a cochain complex (A, d) is of degree 1. The degree of a homogeneous element x is denoted by $|x|$. The symmetric group on n elements is denoted $\mathbb{S}_n$. We follow the Koszul sign rule. That is, the symmetry isomorphism $U \otimes V \xrightarrow{\cong} V \otimes U$ that identifies two graded vector spaces is given on homogeneous elements by $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$. Algebras over operads are always differential graded (dg) and cohomological. We will frequently omit the adjective "dg" and assume it is implicitly understood. The ring of p-adic numbers is denoted $\widehat{\mathbb{Z}_p}$. We shall generally take the perspective that this is the completion of $\mathbb{Z}_{(p)}$ with respect to the p-adic norm. The functor of p-adic de Rham forms $\Omega(-)$ generally depends on a prime p, but to avoid needing to specify this each time, we shall assume that p is fixed.

This is a short article and we do not intend to load it excessively with recollections; so therefore we refer to [16] for the definition of an operad and other basic notions.

2 Preliminaries

In this part, we shall discuss the basic preliminaries. First, we shall discuss E_∞-algebras and why they model spaces. Next, we shall review the basic ideas from rational homotopy theory that we shall need. Then, we shall discuss the different notions of algebra in mixed characteristic and define divided power algebras. Finally, we shall define the homotopy categories of commutative and E_∞-algebras. This last section contains some non-standard material, and is likely the only section the expert reader needs to read.

2.1 E_∞-algebras and Steenrod operations

The free commutative algebra functor is not homotopy invariant in positive or mixed characteristic. The essential problem is that $\mathsf{Com}(n) = \Bbbk$ is not free as a representation of $\mathbb{S}_n$. The traditional way of fixing this is by replacing Com with a weakly equivalent operad $\mathscr{E}$ such that, for each n, the action of $\mathbb{S}_n$ on $\mathscr{E}(n)$ is free. There is some room for choice here, and any such operad is called an E_∞-operad. The precise choice of E_∞ operad we shall use in this article is the Barratt-Eccles operad $\mathscr{E}$. The reader should note that our results will also hold for any other E_∞-operad. For more information about the Barratt-Eccles operad, see [1].

Definition 2.1. The simplicial sets defining the Barratt-Eccles operad in each arity are of the form

$$\mathscr{E}(r)_n = \{(w_0, \ldots, w_n) \in \mathbb{S}_r \times \cdots \times \mathbb{S}_r\}$$

equipped with face and degeneracy maps

$$d_i(w_0, \ldots, w_n) = (w_0, \ldots, w_{i-1}, \hat{w}_i, w_{i+1}, \ldots, w_n)$$
$$s_i(w_0, \ldots, w_n) = (w_0, \ldots, w_{i-1}, w_i, w_i, w_{i+1}, \ldots, w_n).$$

$\mathbb{S}_r$ acts on $\mathscr{E}(n)$ diagonally, that is to say if $\sigma \in \mathbb{S}_n$ and $(w_0, \ldots, w_n) \in \Gamma(n)$ the

$$(w_0, \ldots, w_n) * \sigma = (w_0 * \sigma, \ldots, w_n * \sigma)$$

Finally the compositions are also defined componentwise via the explicit composition law of

$$\gamma : \mathbb{S}(r) \otimes \mathbb{S}(n_1) \otimes \cdots \otimes \mathbb{S}(n_r) \to \mathbb{S}(n_1 + \cdots + n_r)$$
$$(\sigma, \sigma_1, \ldots, \sigma_r) \mapsto \sigma_{n_1 \cdots n_r} \circ (\sigma_1 \times \cdots \times \sigma_r)$$

where $\sigma_{n_1 \cdots n_r}$ is the permutation that acts on $n_1 + \cdots + n_r$ elements, by dividing them into r blocks, the first of length n_1, the second of length n_2 and so on. It then rearranges the blocks according to σ, maintaining the order within each block.

Remark 2.2. As defined above, the Barratt-Eccles operad is an operad in simplicial sets. It becomes an operad in non-negatively graded chain complexes after applying the singular chains functor. When we work in cohomological grading and cochain complexes, it is will be concentrated in non-positive degrees and is unbounded below. In this article, the notation $\mathscr{E}$ shall always refer to the operad in cochain complexes.

Of course, the main reason why the Barratt-Eccles operad is used is that the cochain complex of a space X is an algebra over it (with integral coefficients).

Theorem 2.3. *[1] For any simplicial set X, we have evaluation products $\mathscr{E}(r) \otimes C^*(X)^{\otimes r} \to C^*(X)$ which are functorial in X which give the cochain complex $C^*(X)$ the structure of an algebra over the Barratt-Eccles operad $\mathscr{E}$. In particular, the classical cup-product of cochains is an operation $\mu_0 : C^*(X)^{\otimes 2} \to C^*(X)$ associated to an element $\mu_0 \in \mathscr{E}(2)^0$.*

2.2 Rational homotopy theory

In this section, we review the rational case and explain the connection between E_∞ algebras, rational topological spaces and strictly commutative. We begin by explaining the constructions

Sullivan's A_{PL} functor, which will be our basis for later constructing the p-adic de Rham form functor Ω. In particular, in Proposition 2.4 we shall explain why Sullivan's approach does not work in positive characteristic. Next, we explain the equivalence in approach with that of singular cochains. Next, we shall discuss the rectification of E_∞-algebras with rational coefficients. Finally, we conclude by explaining Cartan's approach to cochain algebras.

2.2.1 Sullivan's approach to rational homotopy theory

In this section, we briefly revise Sullivan's approach to homotopy theory [21]. In general, if R is a commutative ring, we call any functor $\mathsf{sSet} \to \mathsf{CDGA}_R$ a *cochain algebra*. Recall that the Sullivan's *PL-forms functor* $A_{\mathrm{PL}} : \mathsf{sSet} \to \mathsf{CDGA}_\mathbb{Q}$, also called *the rational de Rham forms functor*, is explicitly defined by taking simplicial set maps against the cochain algebra $A_\bullet^*$,

$$A_{\mathrm{PL}}(X) = \mathsf{sSet}(X, A_\bullet),$$

where

$$A_n = A_{\mathrm{PL}}\left(\Delta^n\right) = \frac{\mathrm{Sym}\left(t_0, ..., t_n, dt_0, ..., dt_n\right)}{\left(\sum t_i - 1, \sum dt_i\right)} \cong \mathrm{Sym}\left(t_1, ..., t_n, dt_1, ..., dt_n\right).$$

Here, each t_i is of degree 0, and dt_i is a degree 1 generator identified with $d(t_i)$ by abuse of notation. See [21, 4]. The object $\mathsf{sSet}\left(X, A_\bullet^*\right)$ is a commutative dg-algebra where $\mathsf{sSet}\left(X, A_\bullet^*\right)_k = \mathrm{Hom}_{\mathsf{sSet}}\left(X, \Omega_\bullet^k\right)$ and the differential is induced by the differential $\Omega_\bullet^k \to \Omega_\bullet^{k+1}$. The algebras Ω_n are, in a very precise sense, the polynomial differential forms with rational coefficients on the n-simplex, and gather into a simplicial object $\Omega_\bullet$ in the category $\mathsf{CDGA}_\mathbb{Q}$.

In the case of A_{PL}, cochain algebras satisfy two additional key properties. First is the *Poincaré Lemma*, which asserts that

$$\widetilde{H}^*\left(A_n; \mathbb{Q}\right) = 0.$$

Second is *extendablity*; which asserts that the restriction map $A_{PL}(X) \to A_{PL}(Y)$ is surjective for every inclusion of simplicial sets $Y \subseteq X$. Although the polynomial forms exist over any base ring R, it is essential that $\mathbb{Q} \subseteq R$ for the Poincaré lemma to hold. To prove this, one can observe that

$$A_{\mathrm{PL}}\left(\Delta^n\right) \cong \left(\mathbb{Q}[t] \otimes \mathrm{Sym}\left(dt\right)\right)^{\otimes n},$$

then give an explicit contraction $K : \mathbb{Q}[t] \otimes \mathrm{Sym}\left(dt\right) \xrightarrow{\simeq} \mathbb{Q}$, given by geometric integration, and extend it (non-canonically) as a contraction from the n-fold tensor product to $\mathbb{Q}$. Although there are choices for this extension, there is a choice given by geometric integration which is quite natural. For example, the explicit formulas for Δ^2 can be taken to be

$$K\left(t_j^n dt_j\right) = \frac{1}{n+1} t_j^{n+1}, \quad j = 1, 2,$$

$$K\left(t_1^n t_2^m dt_1 dt_2\right) = \frac{1}{2}\left(\frac{1}{n+1} t_1^{n+1} t_2^m dt_2 + \frac{1}{m+1} t_1^n t_2^{m+1} dt_1\right).$$

Here, we see the fundamental role played by division by n. In positive characteristic, this is impossible to achieve. hat is, if we consider the functor $A_{\mathrm{PL}}^p : \mathsf{sSet} \to \mathsf{CDGA}_{\mathbb{F}_p}$, constructed in the the same manner as A_{PL} but with $\mathbb{F}_p$-coefficents, then for every prime p, the cohomology algebra $\widetilde{H}^*\left(A_n; \mathbb{F}_p\right)$ is non-trivial, see Proposition 2.4 for the precise computation which we learned from José Moreno-Fernández.

Proposition 2.4. *The cohomology of $A_{\mathrm{PL}}^2\left(\Delta^n\right)$ with $\mathbb{F}_2$-coefficients is in bijection with the tuples*

$$\left(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n\right) \in \mathbb{Z}_{\geq 0}^n \times \{0, 1\}^n$$

satisfying

$$\alpha_i \text{ even} \quad \Longrightarrow \quad \beta_i = 0, \qquad \text{and} \qquad \alpha_i \text{ odd} \quad \Longrightarrow \quad \beta_i = 1.$$

For a fixed tuple as above, its cocyle representative is explictly given by

$$t_1^{\alpha_1} \cdots t_n^{\alpha_n}\left(dt_1\right)^{\beta_1} \cdots \left(dt_n\right)^{\beta_n}.$$

Proof. First, we compute the cohomology with $\mathbb{F}_2$-coefficients of $A_{\mathrm{PL}}(\Delta^1)$. Identify $A_{\mathrm{PL}}(\Delta^1) = S(t, dt)$. Applying Leibniz's rule inductively, we find that

$$d\left(t^k\right) = kt^{k-1}dt \quad \text{for all} k.$$

Therefore, the non-trivial cocyles of $A_{\mathrm{PL}}(\Delta^1)$ are all the even powers t^{2k} in degree 0 and all the elements of the form $t^{2k+1}dt$ for $k \geq 0$ in degree 1. By inspection, these cohomology classes are all distinct. Thus,

$$H^n\left(A_{\mathrm{PL}}(\Delta^1);\mathbb{F}_2\right) = \begin{cases} \left[t^{2k}\right] & \forall\ k \geq 0 & \text{in degree 0,} \\ \left[t^{2k+1}dt\right] & \forall\ k \geq 0 & \text{in degree 1.} \end{cases}$$

It is well-known that $A_{\mathrm{PL}}(\Delta^n) \cong A_{\mathrm{PL}}(\Delta^1)^{\otimes n}$, with the following identifications for all $i = 1, ..., n$:

$$t_i = 1 \otimes \cdots \otimes \underbrace{t}_{i} \otimes \cdots \otimes 1, \qquad \text{and} \qquad dt_i = 1 \otimes \cdots \otimes \underbrace{dt}_{i} \otimes \cdots \otimes 1.$$

Since we are working over a field, the Künneth map is an isomorphism, so that

$$H^*\left(A_{\mathrm{PL}}(\Delta^n)\right) \cong H^*\left(A_{\mathrm{PL}}(\Delta^1)^{\otimes n}\right) \cong H^*\left(A_{\mathrm{PL}}(\Delta^1)\right)^{\otimes n}.$$

A straightforward computation gives the cohomology classes mentioned in the statement. $\square$

2.2.2 Comparison between de Rham forms and singular cochains

We next explain the comparison between the A_{PL} functor and the singular cochains $C^*(-,\mathbb{Q})$ functor. The material in this section is essentially due to Sullivan [21], Bousfield-Gugenheim [4] and Mandell [17]. Recall that $C^*(\Delta^*,\mathbb{Q})$ is a simplicial $\mathscr{E}$-algebra, with the $\mathscr{E}$-algebra structure given by Theorem 2.3.

Definition 2.5. Let A^* and B^* be simplicial $\mathscr{E}$-algebras. The *tensor product* $(A \otimes B)^*$ is given by

$$(A \otimes B)^k(\Delta^n) = \bigoplus_{i+j=k} A^i(\Delta^n) \otimes B^j(\Delta^n)$$

This object is equipped with the obvious face and degeneracy maps. The $\mathscr{E}$-algebra structure on $(A \otimes B)^*(\Delta^n)$ is induced from the diagonal on $\mathscr{E}$ in the obvious way.

Proposition 2.6. *[21] Suppose that A^* and B^* are extendable cochain algebras over $\mathbb{Q}$ that both satisfy the Poincaré lemma. Then $(A \otimes B)^*$ also satisfies the Poincaré lemma and is extendable. In particular,*

$$H^*(A \otimes B)(X) = H^*(X,\mathbb{Q})$$

Now one has the following zig-zag of simplicial $\mathscr{E}$-algebras.

$$A_{PL}^*(\Delta^*) \xrightarrow{\mathrm{id} \otimes 1} (A_{PL} \otimes C^*)(\Delta^*) \xleftarrow{1 \otimes \mathrm{id}} C^*(\Delta^*) \tag{1}$$

For all $X \in \mathsf{sSet}$, this extends to a zig-zag of $\mathscr{E}$-algebras by the universal property of simplicial sets

$$A_{PL}^*(X) \xrightarrow{\sim} (A_{PL} \otimes C)^*(X) \xleftarrow{\sim} C^*(X)$$

and by Proposition 2.6, these maps are quasi-isomorphisms.

2.2.3 Rectification

There is a weak equivalence of operads $\phi : \mathscr{E} \xrightarrow{\sim} \mathsf{Com}$, so it is natural to ask whether or not the pair $(\phi^*, \phi_!)$ forms a Quillen equivalence between $\mathscr{E}$-algebras and Com-algebras. If there is, then *rectification* is said to occur. With coefficients in $\mathbb{Q}$, this is indeed the case; see for example [22]. In particular, this implies that, in zero characteristic, that every $\mathscr{E}$-algebra A has a strictly commutative model given by $\phi_!(A)$.

2.2.4 Cartan's approach to cochain algebras

Outside of characteristic zero, it appears to be very difficult to find cochain algebras that both satisfy the Poincaré Lemma and which are extendable. In [8], Cartan extended Sullivan's approach to more general cochain algebras. In particular, he proved the the following generalisation of Theorem 2.6.

Theorem 2.7. *[8] Let R be a commutative ring, X be a simplicial set and $A_\bullet^*$ be a simplicial cochain R-algebra. Let the simplicial cochain R-algebra $Z^k A$ be given by the kernel of the differential $d : A^k \to A^{k+1}$. Suppose further that $\pi_i(A^k)$ and $\pi_i(Z^k A)$ are zero when $i \neq k$. Then one has a natural isomorphism $H^k(A(X)) \cong H^k(X, \pi_k(Z^k A))$. Moreover this isomorphism is multiplicative when the $Z^k A$ are flat R-modules.*

2.3 Algebras over an operad over a ring of positive characteristic

The concept of a divided power algebras was first introduced by Cartan [7] for the commutative operad, and by Fresse [13] for more general operads. In this section, we recall the general definition and, in particular, we explain how to compute the free commutative divided powers algebra on a free module.

Definition 2.8. [13] Let A be dg-module over a commutative unital ring. We say that A is a $\mathscr{P}$-*algebra* if it is an algebra over the monad

$$\mathscr{P}(V) = \bigoplus_{n \geq 0} \left(\mathscr{P}(n) \otimes V^{\otimes n} \right)_{\mathbb{S}_n}.$$

An algebra over the monad

$$\mathscr{P}(V) = \bigoplus_{n \geq 0} \left(\mathscr{P}(n) \otimes V^{\otimes n} \right)^{\mathbb{S}_n},$$

is referred to as a *divided powers $\mathscr{P}$-algebra*.

Remark 2.9. The notion of a $\mathscr{P}$-algebra and a divided powers $\mathscr{P}$-algebra coincide over a field of characteristic 0. With more general coefficent rings however, there are many examples of $\mathscr{P}$-algebras that are not divided powers $\mathscr{P}$-algebras.

We shall mainly be interested in the case $\mathscr{P} = \mathsf{Com}$, so it will be useful to be more explicit in this case. Let R be a commutative unital ring. The cofree conilpotent coalgebra on a graded projective R-module V, also called *tensor coalgebra* on V, is the graded R-module

$$TV = \bigoplus_{k \geq 0} T^k V, \tag{2}$$

where $T^k V = V^{\otimes k}$ for all k, endowed with the deconcatenation coproduct,

$$\Delta[v_1|\cdots|v_n] = \sum_{i=0}^{n} [v_1|\cdots|v_i] \otimes [v_{i+1}|\cdots|v_n].$$

A basis tensor of TV is therefore denoted $[v_1|\cdots|v_n]$ rather than $v_1 \otimes \cdots \otimes v_n$. The direct sum decomposition in (2) is called the *word-length decomposition* of TV, and elements in $T^k V$ are said to be of word-length k. The tensor coalgebra can be endowed with the associative and commutative shuffle product $\circledast$,
explicitly given by

$$[v_1|\cdots|v_p] \circledast [v_{p+1}|\cdots|v_n] = \sum_{\sigma \in S(p,q)} \varepsilon(\sigma) \, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Here, $S(p,q)$ is the set of (p,q)-shuffles, given by those permutations of $p+q$ elements such that

$$\sigma(1) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q),$$

while $\varepsilon(\sigma)$ stands for the Koszul sign associated to the permutation σ. Endowed with the deconcatenation coproduct and the shuffle product, TV is a commutative bialgebra.

There is a natural action of the symmetric group S_n on the word-length n components of TV, given by

$$\sigma \cdot [v_1|\cdots|v_n] = \varepsilon(\sigma) \cdot [v_{\sigma^{-1}(1)}|\cdots|v_{\sigma^{-1}(n)}].$$

For each n, one can form the submodule of S_n-invariants under this action, that is, the submodule generated by those word-length n homogeneous elements x with $\sigma \cdot x = x$ for all $\sigma \in S_n$. Denote by $\Gamma^n V$ this submodule of $T^n V$. Summing over all n, we form a graded submodule of TV,

$$\Gamma(V) = \bigoplus_{n \geq 0} \Gamma^n V.$$

The submodule ΓV happens to be a subalgebra of TV, and it is called the *free commutative divided powers algebra on V*. It comes equipped with set-theoretical maps $\gamma^k : \Gamma V \to \Gamma V$ determined by

$$\gamma^0(v) = 1 \text{ for all } v \in V_{2n},$$

$$\gamma^n(v) = [v|\cdots|v](n \text{ times if } v \text{ is of even degree and } n \geq 1, \text{ and}$$

$$\gamma^n(v) = 0 \text{ if } v \text{ is of odd degree and } n \geq 2.$$

In particular, the following two identities are satisfied on homogeneous elements (the second one only when u is of even degree):

$$\gamma^n(u+v) = \sum_{i=0}^{n} \gamma^i(u)\gamma^{n-i}(v),$$

$$\gamma^i(u)\gamma^j(u) = \binom{i+j}{i}\gamma^{i+j}(u).$$

Intuitively, the element $\gamma^n(u)$ is a replacement of the element $\frac{u^n}{n!}$ whenever it does not make sense to divide by $n!$.

Assume V is freely generated by the homogeneous elements $\{v_i\}$. Then, an R-linear basis of ΓV is explicitly given by elements of the form

$$\gamma^{k_1}(v_1)\gamma^{k_2}(v_2)\cdots\gamma^{k_r}(v_r)$$

for all $r \geq 0$, $k_i \geq 0$, with $k_i \in \{0,1\}$ if $|v_i| = 1$.

Example 2.10. Let $t \geq 1$. A very useful example occurs when V is a free R-module $R^{\otimes t}$ with basis $x_1, \ldots x_t$. In this case ΓV is usually called *divided power polynomial algebra* and denoted $R\langle x_1, \ldots, x_t\rangle$. Explicitly, we have that

$$R\langle x_1, \ldots, x_t\rangle := \bigoplus_{n_1, \ldots, n_t \geq 0} Rx_1^{[n_1]}, \ldots, x_t^{[n_t]}$$

with multiplication is given by

$$x_i^{[n]}x_i^{[m]} = \frac{(n+m)!}{n!m!}x_i^{[n+m]}$$

We also set $x_i = x_i^{[1]}$. Note that $1 = x_1^{[0]}\cdots x_t^{[0]}$. There is an canonical R-algebra map $R\langle x_1, \ldots, x_t\rangle \to R$ sending $x_i^{[n]}$ to zero for $n > 0$. The kernel of this map is denoted $R\langle x_1, \ldots, x_t\rangle_+$

Example 2.11. When $R = \mathbb{F}_p$, as a commutative dg-algebras

$$\mathbb{F}_p\langle x\rangle = \begin{cases} \mathbb{F}_p[x_1, x_2, \ldots]/\left(x_1^p, x_2^p, \ldots\right) & \text{with } |x_k| = k|x|, \text{ when } |x| \text{ is even.} \\ \mathbb{F}_p[x]/\left(x^2\right) & \text{otherwise.} \end{cases}$$

Example 2.12. When $R = \widehat{\mathbb{Z}_p}$, the divided powers algebra $\widehat{\mathbb{Z}_p}\langle x_1, \ldots x_t\rangle$ is a subalgebra of usual polynomial algebra $\mathbb{Q}_p[x_1, \ldots x_t]$ via the injective map

$$\widehat{\mathbb{Z}_p}\langle x_1, \ldots x_t\rangle \hookrightarrow \mathbb{Q}_p[x_1, \ldots x_t]$$

$$x_i^{[n]} \mapsto \frac{1}{n!}x_i^n$$

2.4 The homotopy theory of $\mathcal{E}$-algebras and commutative dg-algebras

In this subsection we shall discuss the existence of model structures on categories of $\mathcal{P}$-algebras and specialise to the cases of $\mathcal{E}$-algebras. The key takeaway of this subsection is that, in this paper, we shall work with the external homotopy category of commutative algebras instead of the naive (internal) one.

2.4.1 The case of E_∞ algebras

One has the following general fact.

Theorem 2.13. *[15] Let $\mathcal{P}$ be a $\mathbb{S}$-split (or cofibrant) operad over a commutative ring R. Then the category of $\mathcal{P}$-algebras over R is a closed model category with quasi-isomorphisms as the weak equivalences and surjective maps as fibrations.*

The Barratt-Eccles operad is $\mathbb{S}$-split. This immediately gives the model structure on E_∞-algebras over $\widehat{\mathbb{Z}_p}$.

Definition 2.14. The model category $\mathcal{E}-\mathsf{alg}$ of E_∞-algebras is the category of algebras over the Barratt-Eccles operad, in dg-modules over $\widehat{\mathbb{Z}_p}$, equipped with the model structure of Theorem 2.13. It has quasi-isomorphisms of chain complexes as weak equivalences and surjective maps as fibrations.

2.4.2 The case of commutative dg-algebras

We have already mentioned that in characteristic 0, the homotopy theory of commutative dg-algebras coincides with that of $\mathcal{E}$-algebras. In positive characteristic the relationship is much more complex. Commutative dg-algebras come with an obvious notion of weak equivalence, that is, algebra maps that are quasi-isomorphisms of cochain complexes. Localising with respect to these maps gives a well-defined homotopy category, which we call the *internal homotopy category*. The main result of [12] shall show that this is the wrong homotopy category to consider when working with spaces. Instead, we shall consider the *external homotopy category*.

Definition 2.15. The *external homotopy category* of commutative algebras is defined by taking the full subcategory of $\mathcal{E}-\mathsf{alg}$ given by $\mathcal{E}$-algebras that are quasi-isomorphic to strictly commutative dg-algebras and localising it at quasi-isomorphisms of $\mathcal{E}$-algebras.

The external homotopy category of commutative algebras is clearly a full subcategory of the homotopy category of $\mathcal{E}$-algebras. It works well for forming constructions such as derived mapping spaces.

3 The de Rham forms over $\widehat{\mathbb{Z}_p}$

We saw in Proposition 2.4, that Sullivan's A_{PL} functor fails to generalise to positive characteristic. This problem can partially be solved by trading the free polynomial algebra appearing in the definition for a free divided powers algebra. The resulting object, $\Omega^*(X)$, has the correct cohomology but is not quasi-isomorphic to the singular cochains on X as an $\mathcal{E}$-algebra. It is however very closely related. We shall see in the next part that it enables us to define and calculate Massey products in situations where this machinery was previously inconvenient, for example, one has Massey products arising in the torsion part of the cohomology.

This section of the paper is broken into four subsections. The first is devoted to defining $\Omega^*(X)$. In the second, we compute the cohomology of this object. In the third, we explain how it is related to the singular cochain algebra. Finally, in the fourth, we explain the universal property defining it.

3.1 The algebra of p-adic de Rham forms

In this subsection, we introduce the key object of this paper - a generalisation of Sullivan PL-forms to the p-adic setting. We show that this generalisation satisfies the Poincaré lemma, but not the extendable condition (as defined in Section 2.2.1). As mentioned in the introduction, a similar object to $\Omega^*(X)$ appears in [8, Section 4].

Definition 3.1. The *p-adic de Rham cochain algebra* $\Omega^*_\bullet$ is a simplicial cochain algebra that has for n-simplices

$$\Omega^*_n = \left(\frac{\widehat{\mathbb{Z}_p}\langle x_0, \dots x_n \rangle \otimes \Lambda(dx_0, \dots, dx_n)}{(x_0 + \dots + x_n - p, dx_0 + \dots dx_n)} \right)^c, \quad |x_i| = 0, \ |dx_i| = 1.$$

Here, the $(-)^c$ indicates that we are taking the closure of this set under an formal interchange of variables

$$x_r \mapsto p - \sum_{i=0}^{j} x_{k_i}$$

$$dx_r \mapsto -\sum_{i=0}^{j} dx_{k_i}$$

for all r and such that the x_{k_i} are all distinct from each other and from x_r.

The differential $d : \Omega^*_n \to \Omega^{*+1}_n$ is determined by the formula

$$d(f) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i$$

for $f \in \Gamma_p(x_0, \dots, x_n) / (x_0 + \dots + x_n - p)$ and then extended by the Leibniz rule. The simplicial structure is defined as follows

$$d^n_i : \Omega^*_n \to \Omega^*_{n+1} : x_k \mapsto \begin{cases} x_k & \text{for } k < i. \\ 0 & \text{for } k = i. \\ x_{k-1} & \text{for } k > i. \end{cases}$$

and

$$s^n_i : \Omega^*_n \to \Omega^*_{n-1} : x_k \mapsto \begin{cases} x_k & \text{for } k < i. \\ x_k + x_{k+1} & \text{for } k = i. \\ x_{k+1} & \text{for } k > i. \end{cases}$$

Example 3.2. The 0-simplices $\Omega^0_\bullet$ are given by

$$\frac{\widehat{\mathbb{Z}_p}\langle x_0 \rangle \otimes \Lambda(dx_0)}{(x_0 - p, dx_0)} = \widehat{\mathbb{Z}_p}[p, \frac{p^2}{2}, \dots \frac{p^k}{k!}, \dots] = \widehat{\mathbb{Z}_p}$$

On the other hand, one has that

$$\frac{\widehat{\mathbb{Z}_p}\langle x_0 \rangle \otimes \Lambda(dx_0)}{(x_0 - 1, dx_0)} = \widehat{\mathbb{Z}_p}[\frac{1}{p}, \frac{1}{2 \cdot p^2}, \dots \frac{1}{k! \cdot p^k}, \dots] = \widehat{\mathbb{Q}_p}.$$

This is why we must impose the condition that $x_0 + \dots + x_n = p$ and cannot imitate the $x_0 + \dots + x_n = 1$ condition from the definition of the algebra of piecewise polynomial forms.

The p-adic de Rham forms cochain complex have one of the two desirable properties of a cochain algebra: they satisfy the Poincaré lemma.

Proposition 3.3. *The simplicial cochain algebra Ω^* satisfies the Poincaré lemma. In other words:*

$$H^i(\Omega^*_n) = \begin{cases} \widehat{\mathbb{Z}_p} & \text{if } i = 0. \\ 0 & \text{otherwise.} \end{cases}$$

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Proof. Observe that one has the following isomorphism of cochain algebras

$$\widehat{\mathbb{Z}_p}\langle x_0, \dots x_n \rangle \otimes \Lambda(dx_0, \dots, dx_n) \cong \left(\widehat{\mathbb{Z}_p}\langle x \rangle \otimes \Lambda(dx) \right)^{\otimes n+1}.$$

Since $\widehat{\mathbb{Z}_p}\langle x \rangle \otimes \Lambda(dx)$ is free as a $\widehat{\mathbb{Z}_p}$-module, we can apply the Künneth theorem to deduce

$$H^* \left(\widehat{\mathbb{Z}_p}\langle x_0, \dots x_n \rangle \otimes \Lambda(dx_0, \dots, dx_n) \right) = H^* \left(\widehat{\mathbb{Z}_p}\langle x \rangle \otimes \Lambda(dx) \right)^{\otimes n+1}.$$

So the problem reduces to computing $H^* \left(\widehat{\mathbb{Z}_p}\langle x \rangle \otimes \Lambda(dx) \right)$. The elements $x^{[i-1]} dx$ form a linear base for the degree 1 part of this algebra. Further, one has $d\left(x^{[i]} \right) = x^{[i-1]} dx$. The conclusion follows. $\qquad \square$

As in Section 2.2.1, we now Kan extend our cochain algebra along the inclusion $\triangle^* \to \mathsf{sSet}$.

Definition 3.4. Let X be a simplicial set and let p be a fixed prime number.. The p-adic de Rham forms on X is the commutative dg-algebra

$$\Omega^*(X) = \mathsf{sSet}\left(X, \Omega_\bullet^* \right).$$

where $\mathsf{sSet}(X, \Omega_\bullet)_k = \mathrm{Hom}_{\mathsf{sSet}}\left(X, \Omega_\bullet^k \right)$ and the differential is induced by the differential $\Omega_\bullet^k \to \Omega_\bullet^{k+1}$.

The main difficulty with this approach is that $\Omega_\bullet^*$ is not an extendable cochain algebra. In subsection 3.2, we shall use Theorem 2.7 to resolve this problem.

Proposition 3.5. *The cochain algebra $\Omega_\bullet^*$ is not extendable.*

Proof. Recall that $\Omega^*\left(\triangle^0 \right)$ is $\widehat{\mathbb{Z}_p}$. It follows that $\Omega^*\left(\partial\triangle^1 \right) = \widehat{\mathbb{Z}_p} \oplus \widehat{\mathbb{Z}_p}$. Consider the element $(1, p) \in \Omega^*\left(\partial\triangle^1 \right)$. It suffices to prove that there does not exist a polynomial $f(x_0, x_1) \in \Omega^*\left(\triangle^1 \right)$ such that $f(0, p) = 1$ and $f(p, 0) = p$.

Indeed, assume towards contradiction that such an f exists. Then, as p is not invertible in $\widehat{\mathbb{Z}_p}$, $f(0, p) = 1$ implies that f has a constant term which is not divisible by p. On the other hand, $f(p, 0) = p$ implies that the constant term of f is divisible by p. We have obtained the desired contradiction. We can conclude that the map $\Omega^*\left(\triangle^1 \right) \to \Omega^*\left(\partial\triangle^1 \right)$ is not surjective and therefore that $\Omega_\bullet^*$ is not extendable. $\qquad \square$

3.1.1 Some examples

To illustrate the definition of $\Omega^*(X)$, we compute some examples for specific topological spaces X. First, we have the most trivial case.

Example 3.6. When X is a standard n-simplex, one has the de Rham forms $\Omega^*\left(\triangle^n \right) = \Omega_n^*$, where Ω_n^* is the algebra defined in Definition 3.1.

Next, we compute the next simplest group of examples, the spheres of various dimension.

Example 3.7. For the usual simplicial model of $S^1 = \triangle^1/\partial\triangle^1$, one has the following: the $\widehat{\mathbb{Z}_p}$-module $\Omega^0\left(S^1 \right) = (x_0 x_1) \oplus \widehat{\mathbb{Z}_p}$, where $(x_0 x_1)$ is the ideal generated by the monomial in

$$\frac{\widehat{\mathbb{Z}_p}\langle x_0, x_1 \rangle}{(x_0 + x_1 - p)}$$

This can also be written, purely in terms of one variable as the ideal generated by $x_0^2 - px_0$. In the classical computation by Sullivan, this ideal would have been generated by $x_0^2 - x_0$. The $\widehat{\mathbb{Z}_p}$-module $\Omega^1\left(S^1 \right)$ is

$$\frac{\widehat{\mathbb{Z}_p}\langle x_0, x_1 \rangle dx_0 \oplus \widehat{\mathbb{Z}_p}\langle x_0, x_1 \rangle dx_1}{(x_0 + x_1 - p, dx_0 + dx_1)} = \widehat{\mathbb{Z}_p}\langle x_0 \rangle dx_0.$$

One can easily compute the cohomology of $\Omega^*\left(S^1 \right)$. One has $H^0\left(\Omega^*\left(S^1 \right) \right) = \widehat{\mathbb{Z}_p}$, which is generated by 1. One therefore also has $H^1\left(\Omega^*\left(S^1 \right) \right) = \widehat{\mathbb{Z}_p}$ which is generated by dx_0.

In general it follows that, for $S^n = \triangle^n / \partial \triangle^n$, one has that

$$\Omega^i \left(S^n \right) = \begin{cases} \widehat{\mathbb{Z}_p} \langle x_0, x_1, \cdots x_{n-1} \rangle dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{n-1} & \text{for } i = n. \\ \left(\{ x_{\sigma(0)} x_{\sigma(1)} \ldots x_{\sigma(i)} dx_{\sigma(i+1)} \wedge \cdots \wedge dx_{\sigma(n)} : \sigma \in \mathbb{S}_{n+1} \} \right) & \text{for } i < n. \end{cases}$$

where $\mathbb{S}_{n+1}$ acts on the set of indices $\{0, 1, \cdots, n\}$ by permutation and we replace x_n with $p - x_0 + \cdots + x_{n-1}$ and dx_n with $-dx_0 + \cdots - dx_{n-1}$. One therefore recovers that $H^n (\Omega^* (S^n)) = \widehat{\mathbb{Z}_p}$ which is generated by $dx_1 \wedge \cdots \wedge dx_{n-1}$.

We conclude this section by computing an example with non-trivial torsion in its cohomology and therefore which would not have been possible to model in Sullivan's framework.

Example 3.8. The space $\mathbb{R}P^2$ has a simplicial model X with nondegenerate simplices given by $X_2 = \{U, V\}$, $X_1 = \{a, b, c\}$ and $X_0 = \{v, w\}$, with face maps as follows

$$\delta_0 U = b, \qquad \delta_1 U = a, \qquad \delta_2 U = c, \qquad \delta_0 V = a, \qquad \delta_1 V = b, \qquad \delta_2 V = c,$$

$$\delta_0 a = w, \qquad \delta_1 a = v, \qquad \delta_0 b = w, \qquad \delta_1 b = v, \qquad \delta_0 c = v, \qquad \delta_1 c = v$$

We therefore compute $\Omega^* (\mathbb{R}P^2)$. One can easily verify that

$$\Omega^2 \left(\mathbb{R}P^2 \right) = \widehat{\mathbb{Z}_p} \langle x_0, x_1 \rangle dx_0 \wedge dx_1 \oplus \widehat{\mathbb{Z}_p} \langle y_0, y_1 \rangle dy_0 \wedge dy_1$$

Next, one wants to compute $\Omega^1 \left(\mathbb{R}P^2 \right)$. Elements contained in this are clearly of the form

$$f = U_0(x_0, x_1, x_2) dx_0 + U_1(x_0, x_1, x_2) dx_0 + U_2(x_0, x_1, x_2) dx_2 +$$
$$V_0(y_0, y_1, y_2) dy_0 + V_1(y_0, y_1, y_2) dy_0 + V_2(y_0, y_1, y_2) dy_2$$

where $U_i, V_i \in \widehat{\mathbb{Z}_p} \langle t_0, t_1, t_2 \rangle$ and must satisfy relations coming from the simplicial structure of X. Firstly $\delta_0 U = \delta_1 V, \delta_1 U = \delta_0 V$ and $\delta_2 U = \delta_2 V$. This implies that

$$U_1(0, t, s) = V_0(t, 0, s), \qquad U_2(0, t, s) = V_2(t, 0, s) \qquad V_1(0, t, s) = U_0(t, 0, s), \qquad V_2(0, t, s) = U_2(t, 0, s)$$

$$U_0(t, s, 0) = V_0(t, s, 0), \qquad U_1(t, s, 0) = V_1(t, s, 0)$$

Lastly, we have the bottom row of relations, which imply that

$$U_0(s, 0, 0) = U_1(0, s, 0)$$

Similarly the elements of $\Omega^0 \left(\mathbb{R}P^2 \right)$ are of the form

$$f(x_0, x_1, x_2) + g(x_0, x_1, x_2)$$

with $f, g \in \widehat{\mathbb{Z}_p} \langle t_0, t_1, t_2 \rangle$ and where

$$f(0, s, t) = g(s, 0, t), \qquad f(s, 0, t) = g(0, s, t), \qquad f(s, t, 0) = g(s, t, 0)$$

and

$$f(s, 0, 0) = f(0, s, 0).$$

3.2 The cohomology of $\Omega^* (X)$

In this section, we compute the cohomology ring of $\Omega^* (X)$ and show that it coincides with the usual cohomology ring of X. The main result is the following theorem.

Theorem 3.9. *Let X be a simplicial set. The cohomology ring of $\Omega^* (X)$ is isomorphic to the singular cohomology of X. In other words, one has a ring isomorphism*

$$H^* \left(\Omega^* (X) \right) \cong H^* \left(X, \widehat{\mathbb{Z}_p} \right).$$

The arguments in this section are very similar to that in [8, Section 4]. The strategy is that to apply Theorem 2.7. In order to do so, it is necessary to compute the homotopy groups $\pi_i(\Omega^k)$ and $\pi_i(\Omega^k)$.

Proposition 3.10. *The homotopy groups of Ω^k are as follows:*

$$\pi_i\left(\Omega^k\right) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{when } i = k \\ 0 & \text{otherwise.} \end{cases}$$

with the generator of $\pi_i\left(\Omega^k\right)$ being $dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{k-1}$.

First, make the auxiliary definition.

$$\overline{\Omega}_n^* = \left(\frac{\widehat{\mathbb{Z}_p}\langle x_0, \ldots x_n \rangle \otimes \Lambda(dx_0, \ldots, dx_n)}{(x_0 + \cdots + x_n - p)} \right)^c, \quad |x_i| = 0, \ |dx_i| = 1.$$

Here, the $(-)^c$ indicates that we are taking the closure of this set under an formal interchange of variables

$$x_r \mapsto p - \sum_{i=0}^{j} x_{k_i}$$

for all r and such that the x_{k_i} are all distinct from each other and from x_r.

Let $N_*(-)$ be the normalised chains functor. The homotopy groups $\pi_i\left(N_*\left(\overline{\Omega}^*\right)\right)$ are as follows.

Lemma 3.11. *The homotopy groups of $N_*\left(\overline{\Omega}^*\right)$ are as follows:*

$$\pi_k\left(N_*\left(\overline{\Omega}^*\right)\right) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{when } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $k > 0$, then consider a k-cycle $\omega(x_0, \ldots x_k) \in \Omega^k$ such that $\partial_i \omega = 0$. We may use the closure condition to rewrite $\omega(x_0, \ldots x_k)$ such that $\partial_i \omega = 0$ in

$$\widehat{\mathbb{Z}_p}\langle x_0, \ldots x_n \rangle \otimes \Lambda(dx_0, \ldots, dx_n)$$

It then follows that the $(k+1)$-chain $\omega(x_1, \ldots x_{k+1})$ is such that $\partial_0 \omega(x_1, \ldots x_{k+1}) = \omega(x_0, \ldots x_k)$ and $\partial_i \omega(x_1, \ldots x_{k+1}) = 0$ for $i > 0$.

When $k = 0$, the chains are

$$\frac{\widehat{\mathbb{Z}_p}[x_0]}{(x_0 - p)}.$$

But the image of the differential is the ideal generated by (x_0). So therefore

$$\pi_0\left(N_*\left(\overline{\Omega}^*\right)\right) = \frac{\widehat{\mathbb{Z}_p}[x_0]}{(x_0, x_0 - p)} = \mathbb{Z}/p\mathbb{Z}.$$

$\square$

Proof of Proposition 3.10. Consider the ideal I_n^* of $\overline{\Omega}_n^*$ generated by $dx_0 + \cdots + dx_n$. One has the relation

$$\Omega_n^* = \overline{\Omega}_n^* / I_n^*.$$

Multiplication by $dx_0 + \cdots + dx_n$ sends $\overline{\Omega}_n^i$ to $\overline{\Omega}_n^{i+1}$. Observe that the kernel of this map is I_n^i. One therefore has an exact sequence of simplical $\widehat{\mathbb{Z}_p}$-modules

$$0 \to I^i \to \overline{\Omega}^i \to I^{i+1} \to 0$$

We therefore have that Ω^i is isomorphic to I^{i+1}. One has $I^0 = 0$, and therefore by induction, one finds that $\pi_i\left(\Omega^k\right) = 0$ when $i \neq k$ and $\pi_k\left(\Omega^k\right) = \mathbb{Z}/p\mathbb{Z}$ with the generator being $dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{k-1}$.

$\square$

Now, since $d \circ d = 0$, one has a short exact sequence

$$0 \to Z^k \Omega \to \Omega^k \to Z^{k+1} \Omega \to 0.$$

where the first map is the inclusion and the last map is surjective because Ω satisfies the Poincaré Lemma.

Again one can consider the long exact sequence in homotopy. First, one observes that $\pi_i \left(Z^k \Omega \right) = 0$ when $i \neq k, k-1$ and therefore one has an exact sequence

$$0 \to \pi_k \left(Z^k \Omega \right) \to \pi_{k-1} \left(Z^{k-1} D \right) \to \pi_{k-1} \left(\Omega^{k-1} \right) \to \pi_{k-1} \left(Z^k \Omega \right) \to 0.$$

This identifies $\pi_k \left(Z^k \Omega \right)$ as a subgroup of $\pi_{k-1} \left(Z^{k-1} \Omega \right)$. A routine computation shows that $\pi_0 \left(Z^0 D \right) = \widehat{\mathbb{Z}_p}$; and then one can show by induction that $\pi_{k-1} \left(Z^{k-1} D \right) \to \pi_{k-1} \left(\Omega^{k-1} \right)$ is surjective, so it follows that $\pi_{k-1} \left(Z^k \Omega \right) = 0$. The induction therefore gives that

$$\pi_k \left(Z^k \Omega \right) = p^k \widehat{\mathbb{Z}_p}.$$

Finally, we observe that there is an isomorphism $H^k \left(X, p^k \widehat{\mathbb{Z}_p} \right) = H^k \left(X, \widehat{\mathbb{Z}_p} \right)$. We phrase the all of the above as a proposition.

Proposition 3.12. *The cohomology ring of* $\Omega^* (X)$ *is isomorphic to the singular cohomology of X. In other words, one has a ring isomorphism*

$$H^* \left(\Omega^* (X) \right) = H^* \left(X, \widehat{\mathbb{Z}_p} \right).$$

Proof. The computation above gives that

$$\pi_k \left(Z^k \Omega \right) = p^k \widehat{\mathbb{Z}_p}.$$

It therefore follows from Theorem 2.7 that $H^* (\Omega^* (X)) = H^* \left(X, \widehat{\mathbb{Z}_p} \right)$. The $Z^k \Omega$ are submodules of the torsion-free $\widehat{\mathbb{Z}_p}$-modules Ω^k. Therefore they are torsion-free modules over a PID and so are flat. It therefore follows from Theorem 2.7 that the cohomology ring is as in the statement. $\square$

Remark 3.13. As in the rational case, one can check that there is a zig-zag of $\mathscr{E}$-algebras.

$$\Omega^* (X) \xrightarrow{i} (C \otimes \Omega)^* (X) \xleftarrow{j} C^* (X).$$

which is induced by left Kan extending the zig-zig

$$\Omega^* \left(\triangle^* \right) \xrightarrow{1 \otimes \mathsf{id}} (C \otimes \Omega)^* \left(\triangle^* \right) \xleftarrow{\mathsf{id} \otimes 1} C^* \left(\triangle^* \right).$$

along $\triangle^* \to \mathsf{sSet}$. However, these maps do not descend to isomorphisms on cohomology. In fact, one can verify that for $X = S^1$, the map $H^1 (1 \otimes \mathsf{id})$ is multiplication by p. In the next subsection, we shall discuss how one can remedy this.

3.3   The relationship between the p-adic de Rham forms and the algebra of singular cochains

In this subsection, we upgrade the result of the previous section by explaining how to interpret $\Omega^* (X)$ as an $\mathscr{E}$-algebra. Given the nonvanishing of the Steenrod operation P^0, it has no chance of generally being weakly equivalent to the singular cochains on X. However we shall show in this section that it is quasi-isomorphic to the following subalgebra of $C^* \left(X, \widehat{\mathbb{Z}_p} \right)$. First, it is necessary to establish some notation.

3.3.1 The p-shifted singular cochains

In this subsubsection, we shall define the p-shifted singular cochains algebras.

Definition 3.14. Let X be a simplicial set. We define the *p-shifted singular cochain algebra* $\mathscr{D}^*\left(X,\widehat{\mathbb{Z}_p}\right)$ to be the following subalgebra of the singular cochains $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$.

$$\mathscr{D}^n(X) = \left\langle p^i\sigma : \text{for } \sigma \in C^n\left(X,\widehat{\mathbb{Z}_p}\right) \text{ and } \begin{cases} i = n & \text{if } d\sigma = 0. \\ i = n+1 & \text{otherwise.} \end{cases} \right\rangle$$

The differential and the $\mathscr{E}$ structure are that induced by those on $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$.

We quickly verify the basic properties of $\mathscr{D}^*(X)$; namely that $\mathscr{D}^*(X)$ is indeed a sub-$\mathscr{E}$-algebra and we compute its cohomology.

Proposition 3.15. *Let X be a simplicial set, then p-shifted singular cochain algebra $\mathscr{D}^*(X)$ is a sub-$\mathscr{E}$-algebra of $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$ and has cohomology given by $H^*\left(X,\widehat{\mathbb{Z}_p}\right)$.*

Proof. The first claim follows from the fact that for every operation $\mu \in \mathscr{E}(r)^k$, the operation μ is linear in each variable. In particular, if $x_i \in \mathscr{D}(X)^{r_1}$ then $x_i = p^{r_1}x_i'$. Therefore $\mu(x_1, x_2, \cdots x_n) = \mu\left(p^{r_1}x_1', p^{r_2}x_2', \cdots p^{r_n}x_n'\right) = p^{r_1+\cdots+r_n}\mu(x_1, x_2, \cdots x_n) \in \mathscr{D}(X)^{r_1+\cdots r_n-k}$. The cohomology of $\mathscr{D}^*(X)$ can be directly computed as

$$\frac{p^n Z^n\left(X,\widehat{\mathbb{Z}_p}\right)}{d\left(p^n B^{n-1}\left(X,\widehat{\mathbb{Z}_p}\right)\right)} = H^*\left(X, p^n\widehat{\mathbb{Z}_p}\right) = H^*\left(X,\widehat{\mathbb{Z}_p}\right).$$

The lemma follows. $\qquad\qquad\square$

Remark 3.16. The underlying cochain complex of the p-shifted singular cochains complex functor can be viewed as $\eta_p\left(C^*\left(X,\widehat{\mathbb{Z}_p}\right)\right)$, where η is the the Berthelot-Ogus-Deligne [2, 9] *décalage* functor. This is the connective cover with respect to the Beilinson t-structure on filtered complexes. In this case, we are considering the filtration given by powers of the ideal (p). In this context, Theorem 3.17 of this article can be compared with Theorem 7.4.7 and Example 7.6.7 of [3], which suggest that these objects should have strictly commutative models.

3.3.2 The equivalence

Now, we are ready to compute the homotopy type of $\Omega^*(X)$.

Theorem 3.17. *For every simplicial set X, there exists a cochain algebra $\mathscr{V}^*$ such that there is a zig-zag of quasi-isomorphisms of $\mathscr{E}$-algebras*

$$\Omega^*(X) \xrightarrow{\ f\ } \mathscr{V} \otimes \Omega^*(X) \xleftarrow{\ g\ } \mathscr{D}^*\left(X,\widehat{\mathbb{Z}_p}\right)$$

Remark 3.18. The same arguments go through for the complex $\mathrm{Gr}(X)$ of [8] if one adjusts the definitions of $\mathscr{D}^*$ and of $\mathscr{V}^*$ appropriately with respect to the (s)-adic filtration.

The tensor product appearing in the statement is that of Definition 2.5. The proof strategy is to construct a zig-zag similar to that of (1). First, we define $\mathscr{V}^*$.

Definition 3.19. Let X be a simplicial set. We define the $\mathscr{V}^*(X)$ to be the following subalgebra of the singular cochains $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$.

$$\mathscr{V}^n(X) = \left\langle p^i\sigma : \text{for } \sigma \in C^n\left(X,\widehat{\mathbb{Z}_p}\right) \text{ and } \begin{cases} i = 1 & \text{if } n > 0 \text{ or } d\sigma \neq 0. \\ i = 0 & \text{if } n = 0 \text{ and } d\sigma = 0 \end{cases} \right\rangle$$

The differential and the $\mathscr{E}$ structure are that induced by those on $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$.

Next, we compute the cohomology of $\mathscr{V} \otimes \Omega(X)$.

Proposition 3.20. *The cohomology of $\mathcal{V} \otimes \Omega(X)$ is $H^*\left(X, \widehat{\mathbb{Z}_p}\right)$.*

Proof. The strategy is to compute both $\pi_i\left((\mathcal{V} \otimes \Omega)^k\right)$ and $\pi_i\left(Z^k\left(\mathcal{V} \otimes \Omega^*\right)\right)$, and then the result will follow by an immediate application of Theorem 2.7. The first step is observe that one has

$$\pi_r\left(N_*\left(\mathcal{V}^k\right)\right) = \begin{cases} \mathbb{F}_p & \text{when } i = 0. \\ 0 & \text{otherwise.} \end{cases}$$

where $\pi_0\left(\mathcal{V}^0\right)$ is generated by 1. The cohomology of $N_*\left((\mathcal{V} \otimes \Omega)_\bullet^k\right)$ can then be directly computed using the Kunneth theorem. In particular one has the following short exact sequence

$$\bigoplus_{i+j=k} \bigoplus_{p+q=r} \pi_p\left(\mathcal{V}^i\right) \otimes \pi_q\left(\Omega^j\right) \to \pi_r\left((\mathcal{V} \otimes \Omega)_\bullet^k\right) \to \bigoplus_{i+j=k} \bigoplus_{p+q=r-1} \mathrm{Tor}_1\left(\pi_p\left(\mathcal{V}^i\right), \pi_q\left(\Omega^j\right)\right)$$

First observe that $\pi_p\left(\mathcal{V}^i\right) = 0$ except when $p = 0$ and $\pi_q\left(\Omega^j\right) = 0$ except when $q = j$. We can therefore deduce that

$$\bigoplus_{i+j=k} \bigoplus_{p+q=r-1} \mathrm{Tor}_1\left(\pi_p\left(\mathcal{V}^i\right), \pi_q\left(\Omega^j\right)\right) = 0$$

We conclude that

$$\pi_i\left((\mathcal{V} \otimes \Omega)^k\right) = \begin{cases} \mathbb{F}_p & \text{when } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Now one has a short exact sequence

$$0 \to Z^k\left(\mathcal{V} \otimes \Omega\right) \to (\mathcal{V} \otimes \Omega)^k \to Z^{k+1}\left(\mathcal{V} \otimes \Omega\right) \to 0.$$

Again one can consider the long exact sequence in homotopy. First, one observes that $\pi_i\left(Z^k\left(\mathcal{V} \otimes \Omega\right)\right) = 0$ when $i \neq k, k-1$ and therefore one has an exact sequence

$$0 \to \pi_k\left(Z^k\left(\mathcal{V} \otimes \Omega\right)\right) \to \pi_{k-1}\left(Z^{k-1}\left(\mathcal{V} \otimes \Omega\right)\right) \to \pi_{k-1}\left((\mathcal{V} \otimes \Omega)^{k-1}\right) \to \pi_{k-1}\left(Z^k\left(\mathcal{V} \otimes \Omega\right)\right) \to 0.$$

This identifies $\pi_k\left(Z^k\left(\mathcal{V} \otimes \Omega\right)\right)$ as a subgroup of $\pi_{k-1}\left(Z^{k-1}\left(\mathcal{V} \otimes \Omega\right)\right)$. Since $\pi_0\left(Z^0\left(\mathcal{V} \otimes \Omega\right)\right) = \widehat{\mathbb{Z}_p}$ and one can show by induction that $\pi_{k-1}\left(Z^{k-1}\left(\mathcal{V} \otimes \Omega\right)\right) \to \pi_{k-1}\left((\mathcal{V} \otimes \Omega)^{k-1}\right)$ is surjective, it follows that $\pi_{k-1}\left(Z^k\left(\mathcal{V} \otimes \Omega\right)\right) = 0$. The induction therefore gives that

$$\pi_k\left(Z^k\left(\mathcal{V} \otimes \Omega\right)\right) = p^k\widehat{\mathbb{Z}_p}.$$

Therefore, since $Z^k\left(\mathcal{V} \otimes \Omega\right)$ is free, by Theorem 2.7, we have that $H^i\left(\mathcal{V} \otimes \Omega(X)\right) = H^i\left(X, p^i\widehat{\mathbb{Z}_p}\right) = H^i\left(X, \widehat{\mathbb{Z}_p}\right)$ as desired. $\qquad \square$

We can now prove our main theorem.

Proof of Theorem 3.17. Observe that there is an obvious inclusion $i : \mathscr{D}^*(\triangle^n) \to \mathcal{V}^*(\triangle^n)$ induces a map of $\mathscr{E}$-algebras

$$f_n : \mathscr{D}^*\left(\triangle^n\right) \to (\mathcal{V} \otimes \Omega)^*\left(\triangle^n\right)$$

$$x \mapsto i(x) \otimes 1$$

and, we also have a homotopy equivalence

$$g_n : \Omega^*\left(\triangle^n\right) \to (\mathcal{V} \otimes \Omega)^*\left(\triangle^n\right)$$

$$x \mapsto 1 \otimes x$$

These maps are both compatible with the simplicial structure on the cochain algebras. For all $X \in \mathsf{sSet}$, this extends to a zig-zag of $\mathscr{E}$-algebras by the universal property of simplicial sets

$$\mathscr{D}^*(X) \xrightarrow{f} (\mathcal{V} \otimes \Omega)^*(X) \xleftarrow{g} \Omega^*(X).$$

and by Proposition 3.20, these maps are quasi-isomorphisms. $\qquad \square$

4 Homotopy invariants

In this section we shall discuss some applications of the p-adic de Rham forms. First, we shall show that they recover the Massey products. Recall that Massey products, first defined in [19] and extensively studied in [5], are secondary operations defined on the homology of differential graded associative algebras. They are a finer invariant than the cohomology ring. For example, they can be used to show that the Borromean rings are non-trivially linked, which cannot be detected using only the cohomological cup product. We shall also discuss the relationship between $\mathbb{Q}$-formality and $\widehat{\mathbb{Z}_p}$-formality.

4.1 Massey products in $\Omega^*(X)$

This section discuss the homotopical applications of $\Omega^*(X)$. We shall show that that it allow us to use the machinery of Massey products in situations where they were previously unavailable, for example, in the torsion part of the cohomology of spaces. We finish this section by giving an example of a space X that is formal over $\mathbb{Q}$ but not over $\widehat{\mathbb{Z}_p}$.

We begin by showing that all traditional Massey products in $A_{PL}(X)$ (that is to say, Massey products in the sense of [19] that are defined over $\mathbb{Q}$) may also be computed using $\Omega^*(X)$.

Proposition 4.1. *Suppose that $\sigma \in H^*(X,\mathbb{Q})$ be the higher Massey product of $\langle x_1, x_2, \ldots, x_n \rangle \in H^*(A_{PL}(X),\mathbb{Q})$. Then there exists an $n > 0$ such that $p^n\sigma \in H^*\left(X,\widehat{\mathbb{Z}_p}\right)$ is the higher Massey product of $\langle p^n x_1, p^n x_2, \ldots, p^n x_n \rangle \in H^*\left(A_{PL}(X),\widehat{\mathbb{Z}_p}\right)$ computed in $\Omega^*(X)$.*

Proof. Let $\{a_{i,j}\}$ be a defining system for a Massey product in $A_{PL}(X)$. The inclusion

$$\widehat{\mathbb{Z}_p}\langle x \rangle \to \mathbb{Q}_p[x]$$

induces an inclusion of $\widehat{\mathbb{Z}_p}$-modules

$$f : \Omega^*(X) \hookrightarrow A_{PL}(X) \otimes \mathbb{Q}_p.$$

given by

$$x_{i_1} \cdots x_{i_n} dx_{j_1} \wedge \cdots \wedge dx_{j_m} \mapsto \frac{1}{p^{n+m}} y_{i_1} \cdots y_{i_n} dy_{j_1} \wedge \cdots \wedge dy_{j_m}.$$

Now for a sufficiently large n, the defining system $\{p^n a_{i,j}\}$ must lie in the image of f. Since f is injective, it then can be pulled back to a defining system for $p^n\sigma$ on $\Omega^*(X)$. $\qquad\square$

One can generalise the notion of Massey products with the same definition but choosing cochains representing the torsion part of the cohomology of a space. This has already been done in some special cases. For an example with moment-angle complexes we refer the reader to [14, Example 3.21]. We expect that our construction generalises this, up to factor, and provides a convenient model for doing computations.

4.2 Formality of $\Omega^*(X)$

Recall that a space X is called $\mathbb{Q}$-*formal* if $A_{PL}(X)$ is quasi-isomorphic to the cohomology of X. We shall say that X is $\widehat{\mathbb{Z}_p}$-*formal*, if $\Omega^*(X)$ is quasi-isomorphic to $H^*(X)$ via a zig-zag of commutative dg-algebras. Formality is an extremely useful property in rational homotopy theory, and we hope that $\widehat{\mathbb{Z}_p}$-formality may have similar applications in future.

The main theorem of this section is the following, which is inspired by a conjecture of Mandell [18].

Theorem 4.2. *Let X be a finite simplicial set such that $A_{PL}(X)$ is formal over $\mathbb{Q}$. For all but finitely many primes, $\Omega^*(X)$ is formal over $\widehat{\mathbb{Z}_p}$ as a dg-commutative dg-algebra.*

Before proving this theorem, it will be convenient to introduce some notation and prove a useful lemma.

Definition 4.3. Let V and W be free dg-modules in $\widehat{\mathbb{Z}_p}$. We define the *mixed symmetric algebra* $\mathrm{MSym}(V_0, V_1)$ to be the smallest free commutative dg-algebra containing both $\mathrm{Sym}(V \oplus W)$ and $\Gamma\mathrm{Sym}(W)$.

Lemma 4.4. *Let X be a simplicial set. Suppose that a cochain $\sigma \in \Omega^*(X)$ is not a cocycle. Then there exists a cocycle c such that $(\sigma + c)^{p^n}$ is divisible by p^n.*

Proof. The noncocycles in $\Omega^*(\triangle^n)$ are easily verified to be of the form $\sigma + c$ for $c = 1, 2 \ldots, p - 1 \in \Omega^0(\triangle^n)$. For the general case, observe that

$$\Omega^*(X) = \mathsf{sSet}\left(X_\bullet, \Omega^*(\triangle^n)\right).$$

The result holds for each $x \in X_\bullet$ so the result must hold in the general case. $\qquad\square$

Proof of Theorem 4.2. Before beginning the proof we briefly summarise the idea behind the proof. One constructs a quasi-free, and therefore cofibrant, replacement of $A_{PL}(X)$ in the category of $CDGA_\mathbb{Q}$ via the step-by-step procedure of [10, Proposition 12.1]. At each step one constructs a quasi-free resolution of $\Omega(X)$ with a map to the cofibrant resolution. Finally; if $A_{PL}(X)$ is formal there is a weak–equivalence from the cofibrant resolution of $A_{PL}(X)$ to its cohomology and one shows that this extends to a map on the quasi-free resolution of $\Omega(X)$ to its cohomology.

For all but finitely many primes the cohomology $H^*\left(X, \widehat{\mathbb{Z}_p}\right)$ is torsion-free and therefore projective. Assume we are working at such a prime. In this case,

$$H^*\left(X, \widehat{\mathbb{Z}_p}\right) \otimes_{\widehat{\mathbb{Z}_p}} \mathbb{Q}_p = H^*\left(X, \mathbb{Q}_p\right)$$

Then, we recall that if $A_{PL}(X)$ is formal, then $A_{PL}(X) \otimes \mathbb{Q}_p$ is formal. Now, the inclusion

$$\widehat{\mathbb{Z}_p}\langle x \rangle \to \mathbb{Q}_p[x]$$

induces an isomorphism

$$\widehat{\mathbb{Z}_p}\langle x \rangle \otimes_{\widehat{\mathbb{Z}_p}} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{Q}_p[x].$$

and therefore one has a isomorphism

$$\Omega^*(X) \otimes_{\widehat{\mathbb{Z}_p}} \mathbb{Q}_p = A_{PL}(X) \otimes \mathbb{Q}_p.$$

given by

$$x_{i_1} \cdots x_{i_n} dx_{j_1} \wedge \cdots \wedge dx_{j_m} \mapsto \frac{1}{p^{n+m}} y_{i_1} \cdots y_{i_n} dy_{j_1} \wedge \cdots \wedge dy_{j_m}.$$

This isomorphism restricts to an inclusion of dg-$\widehat{\mathbb{Z}_p}$-modules

$$\Omega^*(X) \to A_{PL}(X) \otimes \mathbb{Q}_p,$$

which is clearly an quasi-isomorphism after tensoring by $\mathbb{Q}_p$. We shall commence by showing that one can build compatible Sullivan-type models for $A_{PL}(X) \otimes \mathbb{Q}_p$ and $\Omega^*(X)$ as commutative dg-algebras. Since $H^*\left(X, \widehat{\mathbb{Z}_p}\right)$ is free and therefore projective, one has a quasi-isomorphism of dg-$\widehat{\mathbb{Z}_p}$-modules

$$H^*\left(X, \widehat{\mathbb{Z}_p}\right) \to \Omega^*(X).$$

One can then choose a map $H^*\left(X, \mathbb{Q}_p\right) \to A_{PL}(X) \otimes \mathbb{Q}_p$ such that the following diagram commutes.

$$
\begin{array}{ccc}
H^*\left(X, \widehat{\mathbb{Z}_p}\right) & \longrightarrow & \Omega^*(X) \\
{\scriptstyle H^*(X,i)}\Big\downarrow & & \Big\downarrow \\
H^*\left(X, \mathbb{Q}_p\right) & \longrightarrow & A_{PL}(X) \otimes \mathbb{Q}_p.
\end{array}
$$

Here, the map $i : \widehat{\mathbb{Z}_p} \to \mathbb{Q}_p$ is the usual inclusion of a ring into its field of fractions. Next, we follow the next step of the classical procedure for building a Sullivan model by extending this to a map of free commutative dg-$\widehat{\mathbb{Z}_p}$-algebras.

$$
\begin{array}{ccc}
\operatorname{Sym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right)\right) & \xrightarrow{f_0} & \Omega^*(X) \\
\downarrow & & \downarrow \\
\operatorname{Sym}\left(H^*\left(X, \mathbb{Q}_p\right)\right) & \xrightarrow{g_0} & A_{PL}(X) \otimes \mathbb{Q}_p.
\end{array}
$$

The reader should observe that $\ker H^*\left(f_0\right) \otimes \mathbb{Q}_p = \ker H^*\left(g_0\right)$ since $H^*\left(X, \widehat{\mathbb{Z}_p}\right)$ has zero differential. Moreover, these kernels are free since we are working over a PID. Therefore, any basis of cocycles W_1 for $\ker H^*\left(f_0\right)$ is such that $W_1 \otimes \mathbb{Q}_p$ is a basis for $\ker H^*\left(g_0\right)$. Therefore one can extend the differential to

$$
d : V_1 = s W_1 \to W_1 \subset \operatorname{Sym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right)\right)
$$

$$
d : V_1 \otimes \mathbb{Q}_p = s V_1 \otimes \mathbb{Q}_p \to W_1 \otimes \mathbb{Q}_p \subset \operatorname{Sym}\left(H^*\left(X, \mathbb{Q}_p\right)\right)
$$

that kill all surplus cocycles. Now, observe that the map

$$
f_1 : V_1 \to \Omega^*(X)
$$

is defined to be any choice of map such that the following diagram commutes

$$
\begin{array}{ccc}
V_1 & & \\
\downarrow{\scriptstyle d} & \!\!\!\!\!\!\diagdown{\scriptstyle f_1} & \\
W_1 & \xrightarrow{f_0} & \Omega^*(X).
\end{array}
$$

In particular, it follows from Lemma 4.4 that f_1 can be chosen such that for all $v \in V_1$, we have $p^n \mid f_1(v)^{p^n}$. Define $g_1 = f_1 \otimes \mathbb{Q}_p$. By freeness, we can produce a commutative diagram

$$
\begin{array}{ccc}
\left(\operatorname{Sym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right) \oplus V_1\right), \Omega\right) & \xrightarrow{f_1} & \Omega^*(X) \\
\downarrow & & \downarrow \\
\left(\operatorname{Sym}\left(H^*\left(X, \mathbb{Q}_p\right) \oplus V_1 \otimes \mathbb{Q}_p\right), \Omega\right) & \xrightarrow{g_1} & A_{PL}(X) \otimes \mathbb{Q}_p.
\end{array}
$$

This, so far, is precisely as in [10, Proposition 12.1]. Now, we claim that the map

$$
f_1 : \operatorname{Sym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right) \oplus V_1\right) \to \Omega^*(X)
$$

extends uniquely to

$$
\left(\operatorname{MSym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right), V_1\right)\right) \to \Omega^*(X).
$$

The existence of such an extension is equivalent to showing that for all $v \in V_1$, the element $\left(f_1(v)\right)^{p^n}$ is divisible by p^n. This is true since $f_1(v)$ was chosen to satisfy the hypotheses of Lemma 4.4.

$$
\begin{array}{ccc}
\left(\operatorname{MSym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right), V_1\right), d\right) & \xrightarrow{f_1} & \Omega^*(X) \\
\downarrow & & \downarrow \\
\left(\operatorname{Sym}\left(H^*\left(X, \mathbb{Q}_p\right) \oplus \left(V_1 \otimes \mathbb{Q}_p\right)\right), d\right) & \xrightarrow{g_1} & A_{PL}(X) \otimes \mathbb{Q}_p.
\end{array}
$$

It is clear we can iterate this procedure provided that two conditions. Namely, we must show that, if

- the cohomology of $\operatorname{MSym}\left(V_0, \bigoplus_{i=1}^k V_i\right)$ is torsion-free.

- the map $\operatorname{MSym}\left(V_0, \bigoplus_{i=1}^k V_i\right) \xrightarrow{f_k} \operatorname{Sym}\left(\bigoplus_{i=0}^k V_i \otimes \mathbb{Q}_p\right)$ is a $\mathbb{Q}_p$-quasi-isomorphism

for $k = N-1$, then the same pair of conditions hold for $k = N$. The first condition is clearly true for our construction since, by assumption, the cohomology of $\Omega^*(X)$ is torsion-free. For the second condition to hold, it suffices to observe that f_N sends cocycles to cocycles because the divided powers of the V_i for $i \geq 1$ kill all surplus cocycles.

It therefore follows that the map

$$\left(\mathrm{MSym}\left(V_0, \bigoplus_{i=1}^{\infty} V_i\right), d\right) \to \left(\mathrm{Sym}\left(\bigoplus_{i=0}^{\infty} V_i \otimes \mathbb{Q}_p\right), d\right)$$

is a $\mathbb{Q}_p$-quasi-isomorphism. Since $\left(\mathrm{Sym}\left(\bigoplus_{i=0}^{\infty} V_i \otimes \mathbb{Q}_p\right), d\right)$ is cofibrant, there is a quasi-isomorphism $\left(\mathrm{Sym}\left(\bigoplus_{i=0}^{\infty} V_i \otimes \mathbb{Q}_p\right), d\right) \to H^*\left(X, \mathbb{Q}_p\right)$. This restricts to a quasi-isomorphism of commutative dg-algebras

$$\left(\mathrm{MSym}\left(V_0, \bigoplus_{i=1}^{\infty} V_i\right), d\right) \to H^*\left(X, \widehat{\mathbb{Z}_p}\right)$$

which implies $\Omega^*(X)$ is formal as desired. $\qquad\square$

5   The best functorial approximation to commutative algebras

In general, the $\widehat{\mathbb{Z}_p}$-de Rham forms lose some information about the associative structure. In this section, we provide an example showing that this is unavoidable. To do this, we show that there are obstructions to strict commutativity in the E_1-algebra structure of some spaces.

Proposition 5.1. *Let A be an associative algebra with $\widehat{\mathbb{Z}_2}$-coefficients. Suppose that there exist $a, b \in A$ such that the Massey product $m(a, b, a)$ is non-vanishing with $\mathbb{F}_2$-coefficents. Then A is not weakly equivalent to a strictly commutative algebra.*

Proof. Suppose A is weakly equivalent to a strictly commutative algebra C. Within $C \otimes \mathbb{F}_2$ one has

$$m(a, b, a) = u \cup a + a \cup u = 0$$

for some u such that $du = a \cup b = b \cup a$. Massey products are well-known to be invariant under quasi-isomorphism, so $m(a, b, a) = 0$ in A as well. The conclusion follows. $\qquad\square$

Remark 5.2. The indeterminacy of this Massey product can be easily checked to be $a \cup H^{|a|+|b|-1}$.

We can produce examples of spaces with such a Massey product. In order to do so, first observe that, in an arbitrary E_∞-algebra, one has

$$m(a, b, a) = u \cup a + a \cup (u + a \cup_1 b).$$

where we suppose $du = a \cup b$. Since $d(u \cup_1 a) = u \cup a + a \cup u + (a \cup b) \cup_1 a$

$$u \cup a + a \cup (u + a \cup_1 b) = d(u \cup_1 a) + a \cup (a \cup_1 b) + (ab) \cup_1 a$$

By the Hirsch identity, one has $(a \cup b) \cup_1 a = a \cup (b \cup_1 a) + (a \cup_1 a) \cup b$. So

$$d(u \cup_1 a) + a \cup (a \cup_1 b) + (ab) \cup_1 a = d(u \cup_1 a) + a \cup (a \cup_1 b) + a \cup (b \cup_1 a) + (a \cup_1 a) \cup b$$

Finally, one has $d(a \cup_1 a \cup_1 b) = a \cup (a \cup_1 b) + a \cup (b \cup_1 a)$. So, finally, one has obtained

$$m(a, b, a) = d(u \cup_1 a) + d(a \cup_1 a \cup_1 b) + (a \cup_1 a) \cup b.$$

In cohomology, the element $(a \cup_1 a) \cup b$ is precisely $\mathrm{Sq}^{|a|-1}(a) \cup b$. So, in conclusion, we have shown that $\mathrm{Sq}^{|a|-1}(a) \cup b$ is a value of $m(a, b, a)$.

We construct an example of a space with a non-vanishing Massey product of the form $m(a, b, a)$.

Example 5.3. Let a be the generator of $H(K(\mathbb{F}_2,3)$ and b be the generator of $H(K(\mathbb{F}_2,2))$. Let X be the homotopy fibre of the map $K(\mathbb{F}_2,3) \times K(\mathbb{F}_2,2) \to K(\mathbb{F}_2,5)$ induced by the element $a \cup b \in H^5(K(\mathbb{F}_2,3) \times K(\mathbb{F}_2,2))$. We can compute the cohomology ring of X using the fibration

$$\Omega(K(\mathbb{F}_2,5)) \hookrightarrow X \twoheadrightarrow K(\mathbb{F}_2,3) \times K(\mathbb{F}_2,2).$$

where we have used the standard argument that the iterated fibre is the loop space of the base space. By the previous discussion, to show that it suffices to show that:

  a) the element $\mathrm{Sq}^2(a) \cup b$ is nonvanishing in cohomology.

  b) the element $\mathrm{Sq}^2(a) \cup b$ is not contained in $aH^4(X)$.

To start, we run a Serre spectral sequence on the above fibration, which tells us that

$$E_2^{p,q} = H^p\left(K(\mathbb{F}_2,3) \times K(\mathbb{F}_2,2), H^q(K(\mathbb{F}_2,4))\right) \Rightarrow H^{p+q}(X).$$

In the above, we have used the identification $\Omega(K(\mathbb{F}_2,5)) \cong K(\mathbb{F}_2,4)$. The cohomology of $K(\mathbb{F}_2,4)$ is the free Steenrod algebra on one generator σ and, by the Kunneth theorem and since we are working over the field $\mathbb{F}_2$, the cohomology of $K(\mathbb{F}_2,3) \times K(\mathbb{F}_2,2)$ is the tensor product of two free Steenrod algebras each on one generator. We have that $\mathrm{Sq}^2(a) \cup b \in E_2^{7,0}$, therefore to prove $\mathrm{Sq}^2(a) \cup b$ is nonvanishing in cohomology, it therefore suffices to show that $E_2^{7,0}$ it is not hit by any of the higher differentials. It is easy to check that $E_2^{p,1} = E_2^{p,2} = E_2^{p,3} = 0$. So, the first differential that could hit $E^{7,0}$ is d_5. Indeed $d_5(b)$ hits $E_2^{7,0}$, but this must kill $a \cup b \cup b$. We have that $E_6^{1,5} = E_2^{1,5} = 0$, so d_6 does not affect $E^{7,0}$. Finally, we have d_7. The domain is $E_7^{6,0} = H^6(K(\mathbb{F}_2,4)) = \mathbb{F}_2^{\oplus 2}$. The generators are $\mathrm{Sq}^2(\sigma)$ and $\mathrm{Sq}^1\mathrm{Sq}^1(\sigma)$. It is straightforward to determine that these are sent to $\mathrm{Sq}^2(a \cup b)$ and $\mathrm{Sq}^1\mathrm{Sq}^1(a \cup b)$.

Having computed $E_2^{7,0}$, we note that this satisfies only the Adem and Cartan relations. The only relation that applies to $\mathrm{Sq}^2 a \cup b$ is

$$\mathrm{Sq}^2 a \cup b + \mathrm{Sq}^1 a \cup \mathrm{Sq}^1 b + a \cup \mathrm{Sq}^2 b = \mathrm{Sq}^2(a \cup b) = 0$$

Thus we can conclude that $\mathrm{Sq}^2 a \cup b \notin aH^4(X)$. Therefore, the product $m(a,b,a)$ is non-vanishing. It follows from Proposition 5.1 that the E_1-structure on X cannot be rectified.

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