Associative and commutative dg-algebras in positive characteristic

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It is a truth universally acknowledged in mathematics that occasionally one finds oneself at the end of a seminar or conference talk and in want of a question to ask. Luckily, there are a few *clichéd* inquiries one can always fall back on. The default varies a bit by area, but algebra and topology, an old chestnut that can be reliably dug out is the following: *what happens if we do this in positive characteristic?* Sometimes, this question can completely flip both the methods and the answer, as is the case in this article, where we tackle the following problem in homotopy theory recently posed and solved in characteristic zero by a team of European researchers [1]. Along the way will get a whirlwind tour of the role that subtle number-theoretic invariants can play in distinguishing very similar objects in algebraic topology.

Question 1. If two commutative dg-algebras A and B are quasi-isomorphic as associative dg-algebras are they quasi-isomorphic as commutative dg-algebras?

To attack this question, one must first understand it and we begin with a short crash course in homological algebra. We will recall what we need, but as this is a vast subject, we refer the reader to [9] for a more detailed introduction.

Definition 2. A *cochain complex* over a field is a family of vector spaces $A^i : i \in \mathbb{Z}$ and *(linear) differential maps* $d^n : A^n \to A^{n+1}$ such that $d^{n+1} \circ d^n = 0$ for all n, where 0 is the zero map.

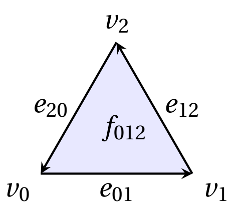
$$\cdots \longrightarrow A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} A^{i+2} \xrightarrow{d} A^{i+3} \xrightarrow{d} A^{i+4} \xrightarrow{d} \cdots$$

A chain complex is the dual notion and has the same definition except the maps go down a degree $\partial_n : A^n \to A^{n-1}$. For notational convenience, we generally suppress the superscript n when the target and source of the map are implicitly understood. A map of cochain complexes $A \to B$ is a collection of maps f_n that commute with the differential. In other words, the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & A^i \arrow[r, "d"] \arrow[d, "f"] & A^{i+1} \arrow[r, "d"] \arrow[d, "f"] & A^{i+2} \arrow[r, "d"] \arrow[d, "f"] & A^{i+3} \arrow[r, "d"] \arrow[d, "f"] & A^{i+4} \arrow[r, "d"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & B^i \arrow[r, "d"] & B^{i+1} \arrow[r, "d"] & B^{i+2} \arrow[r, "d"] & B^{i+3} \arrow[r, "d"] & B^{i+4} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

One elementary way that (co)chain complexes arise in nature is from simplices.

Example 3. Consider the standard 2-simplex (in other words a triangle).



This consists of:

- Three 0-simplices (vertices): v_0, v_1, v_2
- Two 1-simplices (oriented edges): e_{01}, e_{12}, e_{20}
- A single 2-simplex (oriented face): f_{012}

We define the chain groups (which are vector spaces over $\mathbb{R}$):

$$C_2 = \mathbb{R}\langle f_{012}\rangle \cong \mathbb{R}, \quad C_1 = \mathbb{R}\langle e_{01}, e_{12}, e_{20}\rangle \cong \mathbb{R}^3, \quad C_0 = \mathbb{R}\langle v_0, v_1, v_2\rangle \cong \mathbb{R}^3$$

So each vector space has a basis C_i given by the simplices of dimension i. Next, we define boundary maps $\partial_i : C_i \to C_{i-1}$:

$$\begin{aligned}
\partial_2(f_{012}) &= e_{01} + e_{12} + e_{20} \\
\partial_1(e_{01}) &= v_1 - v_0 \\
\partial_1(e_{12}) &= v_2 - v_1 \\
\partial_1(e_{20}) &= v_0 - v_2 \\
\partial_0(v_i) &= 0
\end{aligned}$$

The intuition behind these maps is that given an object, they return a formal sum of the objects forming its boundary. Then:

$$\partial_1 \circ \partial_2(f_{012}) = \partial_1(e_{01} + e_{12} + e_{20}) = (v_1 - v_0) + (v_2 - v_1) + (v_0 - v_2) = 0$$

Thus, we have a chain complex:

$$\cdots \longrightarrow 0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0 \longrightarrow \cdots$$

To obtain a cochain complex, we dualize the chain complex by applying the functor $\mathrm{Hom}(-, \mathbb{R})$ to each term, reversing the arrows:

$$\cdots \longrightarrow 0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \longrightarrow 0 \longrightarrow \cdots$$

The reader somewhat acquainted with differential geometry will already know another example of a cochain complex. If you have a smooth manifold M, the collection of differential forms on M form a cochain complex $\Omega^\bullet(M)$ usually called *the de Rham complex*, where $\Omega^n(M)$ is the vector space of differential n-forms and the differential is the exterior derivative. Both the de Rham forms can be used to compute some interesting global homotopy invariants of manifolds.

Definition 4. The (co)homology of a (co)chain complex A is the following family of abelian groups

$$H^i(A) = \ker d^i / \operatorname{Im} d^{i-1} \qquad \big(\text{resp.} \quad H_i(A) = \ker \partial_i / \operatorname{Im} \partial_{i+1}\big).$$

where $\ker d^i$ is the kernel of d^i and $\operatorname{Im} d^{i-1}$ is the image of d^{i-1}. The group operation is induced by the vector space addition on A. We refer to elements in $\ker d^i$ as *(co)cycles* and elements in $\operatorname{Im} d^{i-1}$ as *(co)boundaries*. A map of cochain complexes $f : A \to B$ induces group homomorphisms

$$H(f^i) : H^i(A) \to H^i(B).$$

Example 5. We can use our chain complex from earlier to compute the homology of a simplex. The kernel of the map ∂_0 is the whole of C_0. The matrix representing ∂_1 is

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

By Gaussian elimination we can compute its image, and see that $\operatorname{Im}\partial_1 = \mathbb{R}[v_1 - v_0, v_2 - v_0]$. So

$$H_0(C) = \ker\partial_1 / \operatorname{Im}\partial_0 = \mathbb{R}[v_0, v_1, v_2]/\mathbb{R}[v_1 - v_0, v_2 - v_0]$$

This is a one-dimensional space, with basis given by the equivalence class of $[v_0]$. So $H_0(C) \cong \mathbb{R}$. The kernel of $\ker\partial_1$ can be also computed from the matrix representation to be $\mathbb{R}[e_{01} + e_{12} + e_{20}]$. So therefore

$$H_1(C) = \ker\partial_2 / \operatorname{Im}\partial_1 = \mathbb{R}[e_{01} + e_{12} + e_{20}]/\mathbb{R}[e_{01} + e_{12} + e_{20}] \cong \{0\}.$$

Finally the kernel of $\ker\partial_2$ is clearly 0, so $H_2(C) = \{0\}$. All higher cohomology groups also vanish.

The remarkable thing about the cohomology groups of $\Omega^\bullet(M)$ is that they are a *homotopy invariant* of the manifold M. In other words, they are preserved by squashing and squeezing (but not cutting) it. A very similar invariant - *singular cohomology* - of locally compact, Hausdorff topological spaces can be produced by probing a topological space with simplices of varying dimensions to produce the *singular cochain complex*.

The obvious question is, then, is the converse true - are manifolds or topological spaces with the same cohomology groups homotopy equivalent? Unfortunately, they are not and it fails spectacularly for for some very simple spaces. For example, $S^1 \times S^1$, a torus, has the same cohomology groups as $S^2 \vee S^1 \vee S^1$, a sphere and two circles glued together at a point, but these spaces are not homotopy equivalent. It would be desirable to have finer invariants, capable of distinguishing more spaces.

As it turns out, the de Rham forms actually have a little more structure: a commutative operation called the *wedge product* that behaves well with respect to the differential. The singular cochains on a topological space X have a similar operation induced by the diagonal map

$$X \to X \times X$$

$$x \mapsto (x, x).$$

Definition 6. An *associative differential graded algebra*, usually called a *dg-algebra*, is a cochain complex A equipped with a binary associative multiplication $- \cup - : A^p \otimes A^q \to A^{p+q}$ and d satisfies the Leibniz rule from differentiation ie.

$$d(x \cup y) = d(x) \cup y + (-1)^{|x|} x \cup d(y).$$

where $|x|$ is the degree of x ie. $|x| = n$ if $x \in A^n$. If the product is also graded commutative[1], ie.

$$x \cup y = (-1)^{|x||y|} y \cup x$$

we call A a *commutative dg-algebra*.

Remark 7. An associative algebra structure on a cochain complex A induces a graded multiplication on the cohomology of A,

$$H^i(A) \otimes H^j(A) \to H^{i+j}(A)$$

on the cohomology groups, turning the direct sum $\bigoplus H^i(A)$ into an associative dg-algebra where the differentials are all zero. Such objects are perhaps better known under the moniker *graded rings*. If the algebra structure is commutative, then the ring will be commutative too.

[1] Note that a commutative algebra is also required to be associative.

Again, a delicate question is how one extracts a homotopy invariant from this algebra structure. To do this, we shall need to introduce the notion of *quasi-isomorphism.*

A quasi-isomorphism between cochain complexes A to B is a map that

a) induces an isomorphism on cohomology.

b) preserves any underlying algebraic structure.

The main issue with extracting a homotopy invariant from quasi-isomorphisms is that they do not necessarily admit inverses.

Example 8. For example, we can consider the following two differential graded algebras[2]. The underlying chain complex structure of the first dg- algebra A is

$$\mathbb{R} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{R}x \xrightarrow{0} \mathbb{R}y \xrightarrow{[y\mapsto x^2]} \mathbb{R}x^2 \xrightarrow{0} \mathbb{R}xy \xrightarrow{[xy\mapsto x^3]} \mathbb{R}x^3 \longrightarrow \cdots$$

The associative dg-algebra structure on A is the polynomial algebra $\mathbb{R}[x,y]/(y^2)$, where $|x| = 2$ and $|y| = 3$. This cochain complex has a differential generated by the rules

$$dx = 0 \qquad dy = x^2$$

and extended via the Leibniz rule. For example,

$$d(xy) = d(x)y + (-1)^2 xd(y) = x^3.$$

Next, we turn to the second dg-algebra B, the underlying chain complex is

$$\mathbb{R} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{R}z \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

The dg-algebra structure on B is $\mathbb{R}[z]/(z^2)$, which is two dimensional. The cohomology of both of these complexes is

$$H^i(A) = H^i(B) = \begin{cases} \mathbb{R} & \text{if } i = 0,2. \\ 0 & \text{otherwise.} \end{cases}$$

Elements of the cohomology group $H^2(A)$ are, by definition, equivalences classes of elements in $\ker d^2 \subseteq \mathbb{R}x$. A *cochain-level basis of* $H^2(A)$, is a set of elements of A^2 in the kernel of d which become a basis of the vector space $H^2(A)$ upon applying the equivalence relation. In this case, $H^2(A)$, which is a one-dimensional vector space, has a cochain-level basis of the form $\{x\}$, and $H^2(B)$ has a cochain-level basis given by $\{z\}$.

There is a quasi-isomorphism of commutative dg-algebras from A to B given as follows.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "0"] \arrow[d, "\mathrm{id}"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R}x \arrow[r, "0"] \arrow[d, "{[x\mapsto z]}"] & \mathbb{R}y \arrow[r, "{[y\mapsto x^2]}"] \arrow[d, "0"] & \mathbb{R}x^2 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R}xy \arrow[r, "{[xy\mapsto x^3]}"] \arrow[d, "0"] & \mathbb{R}x^3 \arrow[r] \arrow[d, "0"] & \cdots \\
\mathbb{R} \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{R}z \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & \cdots
\end{tikzcd}$$

One can check this by computing the induced map on the level of cohomology groups. The map $\mathbb{R} \cong H^0(A) \to H^0(B) \cong \mathbb{R}$ is induced by the far left map in the diagram, which is an isomorphism, so the induced map is also an isomorphism. The map $\mathbb{R}x \cong H^2(A) \to H^2(B) \cong \mathbb{R}z$ is also an isomorphism

[2]The reader who is well versed in rational homotopy theory may recognize A as the minimal model for S^2 and B as its cohomology algebra.

as the cochain level map sends the cochain-level basis $\{x\}$ of the cohomology group $H^2(A)$ to the cochain-level basis $\{z\}$ of $H^2(B)$.

On the other hand, there is no quasi-isomorphism from $g : B \to A$. This is because the induced map on cohomology is a isomorphism, so g must induce an isomorphism $H^2(B) \cong \mathbb{R} \to H^2(A) \cong \mathbb{R}$. For this to be true, on the cochain level, the basis element z has to be sent to ax for some scalar $a \in \mathbb{R}$ and $a \neq 0$. But then, since $z^2 = 0$, one has that $g(0) = g(z^2)$. As g must preserve the algebraic structure, one has $g(z^2) = g(z)^2$. Finally $g(z)^2 = a^2x^2 \neq 0$. So one has that $g(0) \neq 0$. But then g fails to be a linear map as it sends zero to something non-zero!

The relation on dg-algebras of being connected by a **single** quasi-isomorphism $A \to B$ is transitive and reflexive, but as the previous example shows, it is unfortunately not symmetric. Therefore it is not an equivalence relation. But we can extract one from it by "pretending it is" by saying that cochain complexes A and B are quasi-isomorphic as associative dg-algebras if there is a *zig-zag*

$$A = C_1 \xleftarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xleftarrow{f_3} \cdots \xleftarrow{f_{n-2}} C_{n-1} \xrightarrow{f_{n-1}} C_n = B.$$

The key point here is that each of the C_i is an associative dg-algebra and each quasi-isomorphism f_i preserves the algebra structure ie. $f_i(ab) = f_i(a)f_i(b)$. One way to understand this is to view the 'being connected by a single quasi-isomorphism' relation as being formally made symmetric by adding formal inverses to each quasi-isomorphism. The dg-algebras A and B will be said to be *quasi-isomorphic* if there is a zig-zag between them as above.

All homotopy equivalent manifolds or topological spaces have quasi-isomorphic dg-algebras of de Rham forms or singular cochains, respectively. One can therefore determine that two manifolds M and N are not homotopy equivalent if the de Rham complexes or singular cochains on M and N are not quasi-isomorphic. Quasi-isomorphism class is already a much stronger, though more computationally complex, invariant than cohomology ring alone. There are also significant, though too technical to summarise briefly, results in the opposite direction. That is to say, the de Rham complex or singular cochains on M and N being quasi-isomorphic as certain kinds of algebras often implies that M and N are homotopy equivalent as manifolds or topological spaces; see [5, 8] for more details.

So now, a natural question is to try to classify when two associative dg-algebras A and B are quasi-isomorphic.

Example 9. One can consider **quasi-isomorphisms of cochain complexes**. That is, a maps of cochain complexes inducing an isomorphism on cohomology with no multiplicative structure assumed. If we work over a field, there is always a map:

$$A \xrightarrow{\sim} H(A).$$

So, if A has the same cohomology groups as B, we can always find a quasi-isomorphism

$$B \xrightarrow{\sim} H(B) = H(A) \xleftarrow{\sim} A.$$

Thus, two cochain complexes have the same quasi-isomorphism type as cochain complexes, if and only if, they have the same cohomology groups. Therefore, quasi-isomorphisms of cochain complexes, in the absence of additional algebraic structure, are not an interesting notion.

In light of Example 9, one might hope two associative dg-algebras A and B are quasi-isomorphic as associative (or commutative) algebras if and only they have the same cohomology ring. This turns out not to be the case, one can construct concrete counterexamples.

When A and B are **commutative** and the zig-zag consists of **commutative** dg-algebras, figuring out when two dg-algebra are quasi-isomorphic turns out to be a very subtle question, which was elegantly answered by Sullivan [8] with his theory of minimal models, which completely answers the classification problem *rationally*, that is, when the base field is $\mathbb{Q}$. This is an amazing result, because it shows that the rational homotopy type of a topological space is completely determined by a (relatively!) small commutative dg-algebra. It formed part of the work that Sullivan was awarded an Abel prize for in 2022.

This all led Ricardo Campos, Dan Petersen, Daniel Robert-Nicoud, and Felix Wierstra to ask the following question [1]:

Question 10. If two commutative algebras A and B are quasi-isomorphic as **associative** dg-algebras are they quasi-isomorphic as **commutative** dg-algebras?

At first glance, the answer appears to be obviously no. Every zig-zag of commutative algebras

$$A = C_1 \xleftarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xleftarrow{f_3} \cdots \xleftarrow{f_{n-2}} C_{n-1} \xrightarrow{f_{n-1}} C_n = B.$$

is also a zig-zag of associative algebras, as commutative algebras are a special kind of associative algebra.

But, as we shall see in the next example, it is easy to find zig-zags

$$A = C_1 \xleftarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xleftarrow{f_3} \cdots \xleftarrow{f_{n-2}} C_{n-1} \xrightarrow{f_{n-1}} C_n = B.$$

where A and B are commutative, but the algebras C_i are associative but not commutative.

Example 11. There are some very simple examples of non-commutative, associative dg-algebras that are quasi-isomorphic **as associative dg-algebras** to commutative dg-algebras. For a concrete example, consider the following algebra A, with a linear basis which has just five elements $\{x, y, xy, yx, z\}$, with the degrees given by $|x| = |y| = 2$ and $|z| = 3$ (in particular, there are some obvious relations here like $x^2 = y^2 = 0$). We set $dz = xy - yx$.

Now, consider B to be the commutative dg-algebra with a linear basis which has just three elements $\{a, b, ab\}$, with the degrees given by $|a| = |b| = 2$. There is a quasi-isomorphism from A to B as follows.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "\mathrm{id}"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R}x \oplus \mathbb{R}y \arrow[r, "0"] \arrow[d, "{[x\mapsto a] \atop [y\mapsto b]}"] & \mathbb{R}z \arrow[r, "{[z\mapsto xy-yx]}"] \arrow[d, "0"] & \mathbb{R}xy \oplus \mathbb{R}yx \arrow[r, "0"] \arrow[d, "{[xy\mapsto ab] \atop [yx\mapsto -ab]}"] & 0 \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \cdots \\
0 \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{R}a \oplus \mathbb{R}b \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{R}ab \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & \cdots
\end{tikzcd}$$

There are thus many more potential zig-zags in the larger category of associative dg-algebras. Surprisingly, if you work with commutative and associative dg-algebras over a field of characteristic 0, they proved that the answer to Question 10 is actually yes [1]. In other words, if there is a zig-zag between two commutative dg-algebras A and B where some of the C_i are associative but not necessarily commutative, there is always an alternative zig-zag between A and B consisting entirely of commutative dg-algebras. The proof, as one might expect, is very technical and makes a lot of use of a collection of methods informally referred to as the *operadic calculus*.

This still leaves open the case of what happens when one works over a field of characteristic p. This brings us back to the question from the introduction.

Question 12. If two commutative dg-algebras A and B over $\mathbb{F}_p$ are quasi-isomorphic as associative dg-algebras are they quasi-isomorphic as commutative dg-algebras?

Sadly and not completely unexpectedly given the previous discussion, the answer here is no, but how might one construct a counterexample?

The strategy here is that one wants to search for homotopy invariants of commutative dg-algebras that are not homotopy invariants of associative dg-algebras. To build one, we first make the following seemingly trite observation. There is something very bizarre going on with p^{th} powers in characteristic p. Observe that, the Leibniz rule for differentiating tells us that in a commutative algebra A, we have:

$$d(x^p) = px^{p-1}dx = 0.$$

where the second equality holds because we are working over a field of characteristic p. In other words, x^p is always a cocycle. But this relation does not hold in a non-commutative dg-algebra A. Instead, one has

$$d(x^p) = (dx)x^{p-1} + x(dx)x^{p-2} + \cdots + x^{p-1}dx.$$

One cannot rearrange the order of the multiplication in the terms, since A is not assumed to be commutative. So this has no reason to a cocycle and generally will not be.

This suggests the following general strategy. You know that, if two commutative dg-algebras A and B are associative quasi-isomorphic, they have the same cohomology ring and that this ring is even commutative.

In our commutative ring, we may have relations of the form $ab = 0$ for $a, b \in H^\bullet(A)$. Recall the definition of the cohomology groups $H^i(A) = \ker d^i / \operatorname{Im} d^{i-1}$.

This means that, on the level of A, a represents an equivalence class, and one may choose

$$\bar{a} \in \ker d^i \subset A^i.$$

representing it. Such choice may not unique, but given any two choices of representative $\bar{a}$ and $\bar{a}'$, they must be related by the rule

$$\bar{a} - \bar{a}' \in \operatorname{Im} d^{i-1}.$$

So, we may take representatives $\bar{a}, \bar{b} \in A$ ie. for $a, b \in H(A)$ in cohomology. The product $\bar{a}\bar{b}$ in A does not need to be 0 on the nose, but it does need to represent that equivalence class. So $\bar{a}\bar{b} \in \operatorname{Im} d$. It follows that one can find $\bar{c} \in A$ such that

$$\bar{a}\bar{b} = d\bar{c}.$$

Again, $\bar{c}$ is not necessarily unique, but the choice is sufficiently constrained as to still be useful. For example, you can add any cocycle σ to c and one still has

$$d(\bar{c} + \sigma) = \bar{a}\bar{b}$$

but you cannot add a non-cocycle, ie. element such that $d\sigma \neq 0$. Cocycles determine elements of the cohomology, and this will be enough to show that our end result is well-defined in a **quotient** ring of the cohomology.

To produce the final invariant, you can just take the p^{th} power $\bar{c}^p$ and you get a cocycle and therefore an element in the cohomology. This is is not perfectly well defined, but, by keeping track of all our previous choices, you can show, without much difficulty, that it is well-defined as an element in the quotient group

$$\frac{H(A)}{H(A)^p + a^p H(A) + b^p H(A)}.$$

For instance, the $H(A)^p$ accounts for the ambiguity in the choice of $\bar{c}$ as, if we chose $c' = \bar{c} + \sigma$ instead of $\bar{c}$, then $(\bar{c} + \sigma)^p = \bar{c}^p + \sigma^p$. Since σ was a cocycle, its p^{th} power σ^p lives in $H(A)^p$, so in the quotient above $[\bar{c}] = [(\bar{c} + \sigma)^p]$. Similar arguments show that $a^p H(A) + b^p H(A)$ accounts for the ambiguity in the choice of $\bar{a}$ and $\bar{b}$.

So now our strategy becomes clear. We just need to find a pair of commutative dg-algebras A and B in characteristic p with the same cohomology groups, but where this invariant differs. Then we need to find an associative algebra C such that there is a zig-zag of associative weak equivalences

$$A \xleftarrow{\sim} C \xrightarrow{\sim} B. \tag{1}$$

Building C, and ensuring it has the same cohomology ring as A and B, involves some combinatorial trickery. One needs C to have a reasonably complicated commutative cohomology ring, but not be commutative on the cochain level. The trick to getting around this is by working with a type of algebra called *cup-1-algebras* [7], which have two operations. The first is an associative operation $\cup : C \otimes C \to C$, which behaves as one would expect. The trick is that one defines a second operation $\cup_1 : C^p \otimes C^q \to C^{p+q-1}$ such that, when $x, y \in C$ and are cocycles, one has (up to signs) the identity

$$d\left(x \cup_1 y\right) = x \cup y - (-1)^{|x||y|} y \cup x.$$

The cohomology of this will then automatically be commutative as $d\left(x \cup_1 y\right) = 0$ so $[x] \cup [y] - (-1)^{|x||y|} [y] \cup [x]$. When x, y are not cocycles, the behaviour of $\cup_1$ is slightly more complicated but still governed by relations of a similar form.

One uses this to build the algebras A, B, C *around the obstruction:* one starts with free commutative algebras for A, B and a free cup-1-algebra for C and then quotients each by various relations in order to produce a zig-zag as in Diagram 1. This is more of a *guided guessing* than an algorithmic procedure. Finally, one proves that the cohomology rings are all the same and that the maps are quasi-isomorphisms of associative algebras using linear algebra. A concrete example of this procedure can be found in the paper [2, Section 4.2.4]. There, as vector spaces, the commutative dg-algebras A and B are 7 and 14 dimensional respectively, and the associative algebra C ends up having 32 basis elements.

While the example from this article has a more combinatorial flavour than most, the general strategy of finding *higher invariants* that live just above the cohomology has a rich history, going back to Massey [6], who used his eponymous products, using similar vanishing arguments to show that the Borromean rings were pairwise unlinked but cannot be separated. They tend to be used to distinguish pairs of topological spaces. For example, Longoni and Salvatore used them to show that probing spaces with points is not an operation that respects homotopy equivalences [4] and Livernet showed that the Swiss-cheese operad is not quasi-isomorphic to its cohomology [3].

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