SIMPLICIAL ENDOMORPHISM OPERADS AND COALGEBRAS

OISÍN FLYNN-CONNOLLY

ABSTRACT. In recent work of Moreno-Fernandez, Wierstra and the author, a coendomorphism operad in the category of pointed topological spaces endowed with the wedge sum was introduced. In this paper, we construct an analogue completely internal to the category of simplicial sets with the goal of defining simplicial coalgebras. As an application, we show that simplicial n-fold suspensions are coalgebras up to coherent homotopy over the Barratt–Eccles E_n-operad.

1. INTRODUCTION

The little n-cubes operad $\mathbb{D}_n$ was first introduced by J. P. May in his 1972 book *The Geometry of Iterated Loop Spaces* [9], although earlier similar notions appear in the work of Stasheff and Boardman-Vogt. He had noticed that n-fold loop spaces carry a natural monoidal (up to homotopy) structure induced by concatenation of loops. He invented operads in order to capture this underlying structure without reference to the space itself. This approach proved its utility immediately, when he was able to show that any simply connected algebra over $\mathbb{D}_n$ is weakly homotopic to an n-fold loop space, a famous result known as *May's recognition principle*.

This enables the systematic development of *cohomology operations* on iterated loop spaces. For example, it can be shown that the homology of the little n-discs operad is the parameterized Poisson operad Pois_n in chain complexes [3]. This immediately implies that the homology of n-fold loop spaces possesses not just the Pontryagin product induced by the concatenation of loops, but also a binary product of degree $1-n$ called the Browder bracket which is compatible with the concatenation of loops in the sense that the Gerstanhaber relation holds. Integral operations such as the Dyer-Lashof and Kudo-Araki operations arise in this framework as well [5].

Eckmann-Hilton duality suggests that iterated suspensions should possess a parallel theory supporting the development of *homotopy operations*. Moreno-Fernandez, Wierstra, together [10] and with the present author [6] have started the development of such a theory. For each pointed topological space X, we define the *coendomorphism operad* $\mathsf{CoEnd}(X)$. One defines an operad with arity n component

$$\mathsf{CoEnd}_{\mathsf{Top}} := \mathsf{Top}_*(X, X^{\vee n}).$$

Given an operad $\mathscr{P}$, a $\mathscr{P}$-coalgebra is defined to be a pair (X,ϕ) where X is a space and ϕ is an operadic morphism $\mathscr{P} \to \mathsf{CoEnd}(X)$. An analogue of May's recognition principle holds: the $\mathbb{D}_n$-coalgebras are, up to homotopy, precisely the n-fold suspensions.

So far, this theory has been developed in topological spaces. However, modern homotopy theory is most effectively phrased in simplicial terms, and one would like a tractable theory of combinatorial coendomorphism operads and coalgebras internal to $\mathsf{Set}_\triangle$ to provide a more convenient context for studying homotopy operations. The theory does not extend as naively as one might hope, as the wedge sum of Kan complexes is not necessarily a Kan complex. As a consequence of this, the obvious choice, the operad defined in arity n by $\mathsf{Set}_\triangle(X, X^{\vee n})$ is not the same as $\mathsf{Top}_*(X, X^{\vee n})$ in the common homotopy category of topological spaces and simplicial sets. For example, the simplicial set $\mathsf{Set}_\triangle\left(S^1, (S^1)^{\vee n}\right)$ contains n disjoint points, one for each copy of S^1 in the wedge sum. In contrast, $\pi_0\left(\mathsf{Top}(S^1, (S^1)^{\vee n})\right)$ is the amalgamated product $\mathbb{Z}^{*n}$ since it includes the pinch map $S^1 \to S^1 \vee S^1$.

There are two approaches to obtaining a valid coendomorphism operad in the category of simplicial sets.

2020 *Mathematics Subject Classification.* 55P48, 18M85.
Key words and phrases. operads, coalgebras, simplicial sets.

(1) One can pass from X to $|X|$, form $\mathsf{CoEnd}_{\mathsf{Top}}(|X|)$, and return to simplicial sets via $\mathsf{Sing}_\bullet$. This produces a very geometrically transparent operad, but it is very large, typically uncountable even for finite X. Crucially it is not purely combinatorial or internal to $\mathsf{Set}_\triangle$.
(2) By applying Kan's Ex^∞ functor, one obtains a coendomorphism operad $\mathsf{CoEnd}_{\mathsf{Set}_\triangle}(X)$. The advantages of $\mathsf{CoEnd}_{\mathsf{Set}_\triangle}(X)$ directly mirror those of the Ex^∞ functor itself: it is functorial, combinatorially tractable, and remains entirely internal to $\mathsf{Set}_\triangle$, while still preserving the correct homotopical behavior.

We briefly remark that one cannot produce a coendomorphism operad dually by subdividing directly as the Ex^∞-functor does not have a left adjoint sd^∞, although each finite stage Ex^n does. The main contribution of this paper is the definition of $\mathsf{CoEnd}_{\mathsf{Set}_\triangle}(X)$ and the proof that it is indeed an operad. For convenience, we state this as a theorem.

Theorem A. The coendomorphism operad $\mathsf{CoEnd}_{\mathsf{Set}_\triangle}(X)$ is an operad in simplicial sets.

We then show that the two constructions above agree up to homotopy. This justifies that our model is the correct one to simplicially model the topological coendomorphism operad.

Theorem B. For finite simplicial sets X, the coendomorphism operads $\mathsf{Sing}_\bullet\big(\mathsf{CoEnd}_{\mathsf{Top}}(X)\big)$ and $\mathsf{CoEnd}_{\mathsf{Set}_\triangle}(X)$ listed above are weakly equivalent.

Finally, this framework permits a definition of coalgebras internal to simplicial sets. Using model-categorical arguments, we establish the following.

Theorem C. Let X be a finite simplicial set, then the n-fold simplicial suspension $\Sigma^n X$ has the structure of an E_n-coalgebra.

Notation and conventions. All topological spaces are compactly generated and Hausdorff. We will refer to the monoidal category of such spaces equipped with the Kelley product as Top. The symmetric group on n letters is denoted $\mathbb{S}_n$. Our references are as follows: for operads [8]; simplicial sets [7]; for the definition of simplicial suspensions and wedge sums [4]; for the Barratt-Eccles operad [1]; for the Boardman-Vogt resolution (the W-construction) of operads [2].

The structure of this article. This paper has the following structure. First we recall some preliminaries on topological coalgebras and Kan's Ex^∞-functor. Section 3 contains our main results: we construct a coendomorphism operad in simplicial sets. We conclude by proving that simplicial suspensions are E_n-coalgebras.

Acknowledgements: The author thanks Felix Wierstra for useful discussions and guidance.

2. Preliminaries

In this section, we collect some of the prerequisites for understanding this paper. First we recall the theory of coalgebras in topological spaces. Then we describe Kan's Ex^∞-functor, a fibrant replacement functor in the Quillen model category of simplicial sets.

2.1. Coalgebras in topological spaces.

In [6, Definition 2.14], the authors show that one can define a *coalgebra over an operad* in pointed topological spaces. There is a similar, but not equivalent, notion in the category of vector spaces over a fixed field given in [8, Subsection 5.2.17]. We summarise this below.

Definition 2.1. Let X be a pointed topological space. The *topological coendomorphism operad* $\mathsf{CoEnd}_{\mathsf{Top}}(X)$ has arity r component

$$\mathsf{CoEnd}_{\mathsf{Top}}(X)(r) := \mathsf{Top}_*(X, X^{\vee r})$$

For $r = 0$, set $\mathsf{CoEnd}_{\mathsf{Top}}(X)(0) = \mathsf{Top}_*(X, *) = *$. The operadic composition maps are defined by

$$\gamma : \mathsf{CoEnd}_{\mathsf{Top}}(X)(r) \otimes \mathsf{CoEnd}_{\mathsf{Top}}(X)(n_1) \otimes \cdots \otimes \mathsf{CoEnd}_{\mathsf{Top}}(X)(n_r) \to \mathsf{CoEnd}_{\mathsf{Top}}(X)(n_1 + \cdots + n_r)$$

$$(f, f_1, \cdots, f_r) \mapsto (f_1 \vee \cdots \vee f_r) \circ f.$$

The symmetric group action permutes the wedge factors in the output.

Remark 2.2. Note that $\mathsf{CoEnd}_{\mathsf{Top}}(X)$ is naturally pointed. We will normally choose to ignore this extra structure, and will regard $\mathsf{CoEnd}_{\mathsf{Top}}(X)$ as unpointed for the rest of this report.

This immediately allows us to define a coalgebra as an algebra over the coendomorphism operad.

Definition 2.3. Let $\mathscr{P}$ be an (unpointed) operad in the category of topological spaces. A *$\mathscr{P}$-coalgebra* is a pointed space X along with an (unpointed) morphism of operads

$$\Delta : \mathscr{P} \to \mathsf{CoEnd}_{\mathsf{Top}}(X)$$

In this framework, one can prove the following result.

Theorem 2.4. [6, Theorem 2.1] *Let $\Sigma^n X$ be the n-fold suspension of a pointed space X. Then there is a natural map of operads*

$$\Delta : \mathbb{D}_n \to \mathsf{CoEnd}_{\mathsf{Top}}(\Sigma^n X)$$

which encodes the homotopy coassociativity and homotopy cocommutativity of the pinch map. Otherwise said, n-fold suspensions are coalgebras over the little n-discs operad. Furthermore, for any based map $X \to Y$, the induced map $\Sigma^n X \to \Sigma^n Y$ extends to a morphism of $\mathbb{D}_n$-coalgebras.

2.2. **Kan's Ex^∞ functor.** Not all objects are fibrant in the classical Quillen model structure on the category of simplicial sets. Kan introduced the fibrant replacement functor Ex^∞ which computes replacements via the combinatorial process of barycentric subdivision. For more details, we refer the reader to [7, Chapter III].

Definition 2.5. Recall that the nondegenerate simplices of the standard n-simplex $\triangle^n$ are exactly the increasing injections $[m] \to [n]$ with $0 \le m \le n$. These are in one-to-one correspondence with the subsets of $\{0, 1, \ldots, n\}$ of cardinality $m+1$ and thus form a poset under inclusion which we denote $P\triangle^n$. We define the *simplical subdivision* of $\triangle^n$ to be

$$\mathrm{sd}\,\triangle^n := \mathscr{N}(P\triangle^n)$$

where $\mathscr{N}$ is the nerve of the poset (regarded as a small category with morphisms given by inclusions).

Lemma 2.6. [7, Lemma III.4.1] *On the level of geometric realizations, there is a homeomorphism $f : |\mathrm{sd}\,\triangle^n| \xrightarrow{\sim} |\triangle^n|$.*

The notion of subdivision can be extended to any simplicial set, not just the standard simplices. This extension makes use of the notion of a *simplex category*, which we shall introduce next.

Definition 2.7. The *simplex category* $\triangle \downarrow X$ of a simplicial set X, has for objects all simplicial maps $\sigma : \triangle^n \to X$ and has for morphisms, the commutative diagrams of the form

$$\begin{tikzcd} \triangle^n \arrow[r, "\sigma"] \arrow[d, "\theta^*"] & X \\ \triangle^m \arrow[ur, "\tau"] & \end{tikzcd}$$

where θ^* is induced by a unique ordinal map $\theta : [m] \to [n]$.

Definition 2.8. Let X be a simplicial set. The *subdivision* $\mathrm{sd}\,X$ of X is defined to be the simplicial set

$$\mathrm{sd}\,X = \lim_{\triangle^n \to X} \mathrm{sd}\,\triangle^n$$

with the limit indexed by the simplex category of X.

Definition 2.9. Let X be a simplicial set. There is a natural map $v_{\triangle^n} : \mathrm{sd}\,\triangle^n \to \triangle^n$ induced by the map of posets $P\triangle^n \to [n]$ given by

$$[v_0, v_1, \ldots, v_k] \mapsto v_k.$$

The *last vertex map* $v_X : \mathrm{sd}\,X \to X$ is

$$v_X = \lim_{\triangle^n \to X} v_{\triangle^n}$$

with the limit indexed by the simplex category of X.

We define the Ex functor to be the right adjoint of the sd functor.

Definition 2.10. For any simplicial set X we define

$$\mathrm{Ex}(X)_n := \mathrm{Set}_\triangle(\mathrm{sd}\,\triangle^n, X)$$

Definition 2.11. We have a morphism $\mu_X : X \to \mathrm{Ex}(X)$ which is adjoint to the last vertex map. Thus we obtain a diagram

$$X \longrightarrow \mathrm{Ex}(X) \longrightarrow \mathrm{Ex}^2(X) \longrightarrow \cdots$$

The colimit of this diagram is denoted $\mathrm{Ex}^\infty(X)$.

The key properties of the Ex^∞-functor are as follows.

Theorem 2.12. [7, Theorem 4.8] *Let X be a simplicial set. Then:*

(1) Ex^∞*(X) is a Kan complex.*
(2) The canonical map $\eta_X : X \to \mathrm{Ex}^\infty(X)$ is an injective weak homotopy equivalence.
(3) Ex^∞ *preserves Kan fibrations.*
(4) Ex^∞ *preserves finite limits.*

3. Coalgebras in simplicial sets

This section contains the main result of this article. We shall first construct the simplicial coendomorphism operad, then show that it has the correct cohomology type and finally generalise [6, Proposition 2.23], and that n-fold suspensions are coalgebras are coalgebras up to coherent homotopy over the Barratt-Eccles operad.

3.1. **The simplicial coendomorphism operad.** In this section, we wish to extend the notion of coalgebras to the category of simplicial sets. As in topological spaces [6], we are going to do this by defining the notion of a coendomorphism operad. As discussed in the introduction, the operad defined in arity n by $\mathrm{Set}_\triangle(X, X^{\vee n})$ does not have the correct homotopy type due to $X^{\vee n}$ not being a Kan complex even when X is.

This hints at the underlying problem. As we have seen throughout this report, not all simplicial sets are Kan complexes. Thus, not all maps in the homotopy category exist between all pairs of objects in the model. To ensure that they do we must take a fibrant replacement of $X^{\vee n}$. To ensure things remain as combinatorially tractable as possible, we shall use Kan's Ex^∞ functor for this task. The underlying $\mathbb{S}$-module of the desired operad is very easy to describe and we can do this immediately.

Definition 3.1. We define the *simplicial coendomorphism* $\mathbb{S}$*-module* in arity r to be

$$\mathrm{CoEnd}_{\mathrm{Set}_\triangle}(X)(r) := \mathrm{Set}_\triangle(X, \mathrm{Ex}^\infty(X^{\vee r})).$$

Each $\sigma \in \mathbb{S}_r$ induces a map $\sigma^* : X^{\vee r} \to X^{\vee r}$, by permutation of the factors of the wedge sum. Then the symmetric action of the $\mathbb{S}$-module is given by the maps

$$- * \sigma : \mathrm{CoEnd}_{\mathrm{Set}_\triangle}(X)(r) \to \mathrm{CoEnd}_{\mathrm{Set}_\triangle}(X)(r)$$
$$f \mapsto \sigma^* \circ f.$$

Remark 3.2. It is obvious that $- * \sigma$ is a bona fide simplicial map because the degeneracy and face maps of the simplicial mapping space act only on the domain of a n-simplex $f : X \times \triangle^m \to \mathrm{Ex}^\infty(X^{\vee r})$ and not on the codomain.

The next few pages consist of defining the operadic composition maps. We start by recalling some notation.

Observation 3.3. Recall from Section 1 that $\mathrm{Ex}^\infty(X)$ is defined as the colimit of the following chain of injective weak homotopy equivalences

$$X \xrightarrow{\sim} \mathrm{Ex}(X) \xrightarrow{\sim} \mathrm{Ex}^2(X) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathrm{Ex}^i(X) \xrightarrow{\sim} \cdots$$

This implies that for all $x \in \mathrm{Ex}^\infty(X)$ there exists an $N > 0$ such that $x \in \mathrm{Ex}^n(X)$ for all $n > N$. Of course, we are implicitly identifying each $\mathrm{Ex}^n(X)$ with its image in $\mathrm{Ex}^\infty(X)$, where they form an exhaustive filtration.

Definition 3.4. Let X be a simplicial set with only finitely many non-degenerate simplices, and let f be an n-simplex of $\mathrm{CoEnd}_{\mathrm{Set}_\Delta}(X)(r)$. In other words,

$$f \in \mathrm{Set}_\Delta(X, \mathrm{Ex}^\infty(X^{\vee r}))_n.$$

By the definition of simplicial mapping sets, f is a simplicial function $X \times \Delta^m \to \mathrm{Ex}^\infty(X^{\vee r})$. Following Observation 3.3, we can associate an integer N_σ to every simplex $\sigma \in X \times \Delta^m$; this being the smallest N such that $f(\sigma) \in \mathrm{Ex}^N(X^{\vee r})$. We define N_f to be the integer $\max\{N_\sigma\}_{\sigma \in X\times\Delta^m}$.

Remark 3.5. The integer N_f is well-defined because $X \times \Delta^m$, the domain of f, has only finitely many non-degenerate simplices.

Remark 3.6. It is easy to check the following three properties of N_f.

- f factors through $\mathrm{Ex}^{N_f}(X^{\vee r})$.
- N_f is the smallest integer with this property.
- For all $N \geq N_f$, f factors through $\mathrm{Ex}^N(X^{\vee r})$.

Our definition of the coendomorphism operad will make heavy use of the adjunction between Ex and sd. For ease of reading, we shall introduce two pieces of helpful notation.

Notation 3.7. Let $f \in \mathrm{Set}_\Delta(\mathrm{sd}^N(X\times\Delta^m), (X^{\vee r}))$ for $N > 0$. This is adjoint to $f^c \in \mathrm{Set}_\Delta((X\times\Delta^m), \mathrm{Ex}^m(X^{\vee r}))$. Now f^c uniquely extends to an element of $\mathrm{Set}_\Delta((X \times \Delta^m), \mathrm{Ex}^\infty(X^{\vee r}))$ which is the same thing as $\mathrm{Set}_\Delta(X, \mathrm{Ex}^\infty(X^{\vee r}))_n$. We shall denote this element as $\overline{f}$.

Notation 3.8. Let $f \in \mathrm{Set}_\Delta(X, \mathrm{Ex}^\infty(X^{\vee r}))_m$. Then it follows from Remark 3.5 that for all $N \geq N_f$, there is a unique element, which we shall denote (f, N), of $\mathrm{Set}_\Delta(\mathrm{sd}^N(X \times \Delta^m), X^{\vee r})$, such that $\overline{(f,N)} = f$.

Having concluded the preliminaries we are now in a position to define the composition maps. Observe that as the subdivision functor is a left adjoint, it preserves colimits. In particular, it commutes with wedge sums.

Definition 3.9. Let $f \in \mathrm{CoEnd}_{\mathrm{Set}_\Delta}(X)(r)_m$ and $f_i \in \mathrm{CoEnd}_{\mathrm{Set}_\Delta}(X)(n_i)_m$ for $1 \leq i \leq r$. We define the composition map

$$\gamma : \mathrm{CoEnd}_{\mathrm{Set}_\Delta}(X)(r) \times \mathrm{CoEnd}_{\mathrm{Set}_\Delta}(X)(n_1) \times \cdots \mathrm{CoEnd}_{\mathrm{Set}_\Delta}(X)(n_r) \to \mathrm{CoEnd}_{\mathrm{Set}_\Delta}(X)(n_1 + \cdots + n_r)$$

to be $\overline{F}$ where F is the map

$$\begin{aligned} F : \mathrm{sd}^{N+N_f}(X \times \Delta^m) &\xrightarrow{\delta_{\mathrm{sd}^{N_f}(X\times\Delta^m)}} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \Delta^m) \times \mathrm{sd}^{N_f}(X \times \Delta^m)) \\ &\xrightarrow{\mathrm{sd}^N(\mathrm{id}\times\mathrm{sd}^{N_f}(\pi_2))} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \Delta^m) \times \mathrm{sd}^{N_f}(\Delta^m)) \xrightarrow{a} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \Delta^m) \times \Delta^m) \\ &\xrightarrow{(f,N_f)} \mathrm{sd}^N(X^{\vee r} \times \Delta^m)) \xrightarrow{b} \mathrm{sd}^N(X \times \Delta^m)^{\vee r} \xrightarrow{(f_1,N)\vee\cdots\vee(f_r,N)} X^{\vee n_1+\cdots n_r} \end{aligned}$$

where:

- N is the integer $\max(N_{f_1}, \ldots, N_{f_r})$;
- $\delta_{\mathrm{sd}^{N_f}(X\times\Delta^m)} : \mathrm{sd}^{N_f}(X \times \Delta^m) \to \mathrm{sd}^{N_f}(X \times \Delta^m) \times \mathrm{sd}^{N_f}(X \times \Delta^m)$ is the diagonal map;
- $\pi_2 : X \times \Delta^m \to \Delta^m$ is the projection;
- $a : \mathrm{sd}^{N_f}(\Delta^m)) \to \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \Delta^m) \times \Delta^m)$ is the map $\mathrm{sd}^N(\mathrm{id} \times v^{(N_f)}_{\Delta^m})$ where $v^{(N_f)}_{\Delta^m} := v_{\Delta^m} \circ \cdots \circ v_{\mathrm{sd}^{N_f-1}\Delta^m}$ and $v_Z : \mathrm{sd}\, Z \to Z$ is the last vertex map;
- b is an isomorphism, as $\times$ is distributive over the wedge sum, and the wedge sum commutes with subdivision.

We need to check that the definition above gives rise to well-defined operad. We phrase this result as a theorem.

Theorem 3.10. *Let X be a simplicial set with finitely many non-degenerate simplices. Then the composition maps of Definition 3.9 induce an operad structure on the $\mathbb{S}$-module* $\mathrm{CoEnd}_{\mathrm{Set}_\Delta}(X)$.

Before proving this theorem, we wish to make two useful remarks and introduce a final piece of notation.

Remark 3.11. Our first remark concerns the relationship between (f,N) and (f,M) for $M > N \geq N_f$. From the definition of Ex we see that, for all simplicial sets Z and Z', the simplicial morphism $\mathrm{Set}_\triangle(\nu_Z, Z')$ is adjoint to $\mathrm{Set}_\triangle(Z,\mu_{Z'})$, where both

$$\mu_Z : Z \to \mathrm{Ex}(Z).$$

$$\nu_Z : \mathrm{sd}\, Z \to Z.$$

are the maps induced by the last vertex map. Thus we have the relation

$$(f,N)\circ \nu_{\mathrm{sd}^N(X\times\triangle^m)} = (f,N+1).$$

for all $N \geq N_f$ and its obvious extension by induction. A second useful well-known result about ν_Z is that the following diagram commutes

$$\begin{tikzcd} \mathrm{sd}\, Z \arrow[r, "\nu_Z"] \arrow[d, "\mathrm{sd}\, f"] & Z \arrow[d, "f"] \\ \mathrm{sd}\, Z' \arrow[r, "\nu_{Z'}"] & Z'. \end{tikzcd} \tag{1}$$

Notation 3.12. We define $\nu_Z^{(k)} := \nu_Z \circ \cdots \circ \nu_{\mathrm{sd}^{k-1} Z}$.

Remark 3.13. Another useful thing is to note that we can replace N_f in the definition of F with any integer $K \geq N_f$, and F will not change. To see why, call this new map $F(K)$, and then observe, with the help of Diagram 1, that $F(K) = F \circ \nu^{(K-N_f)}_{\mathrm{sd}^{N_f}(X\times\triangle^m)}$. By our previous remark

$$\overline{F\circ \nu^{(K-N_f)}_{\mathrm{sd}^{N_f}(X\times\triangle^m)}} = \overline{F}.$$

Similarly, if we replace N in the definition with a larger integer K', the function F in Definition 3.9 will become another function which we will call $F(K')$. It once again follows from Remark 3.11 and Diagram 1 that this function will be related to F by the identity

$$f(K') = F\circ \nu_Z^{(K'-N)},$$

and so we can also replace N with any larger integer in Definition 3.9 without changing the operad structure.

Theorem 3.10. We need to verify that this defines an operad, starting with the associativity axiom. So we wish to show that

$$\gamma(\gamma(f,f_1,\ldots,f_r), f_{1,1},\ldots,f_{r,n_r}) = \gamma(f,\gamma(f_1,f_{1,1},\ldots f_{1,n_1}),\ldots,\gamma(f_r,f_{r,1},\ldots f_{r,n_r}))$$

for all $f \in \mathrm{CoEnd}_{\mathrm{Set}_\triangle}(X)(r)_m$, $f_i \in \mathrm{CoEnd}_{\mathrm{Set}_\triangle}(X)(n_i)_m$ and $f_{i,j} \in \mathrm{CoEnd}_{\mathrm{Set}_\triangle}(X)(n_{i,j})_m$. Expanding the left hand side of this we obtain

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij},M))\circ(\bigvee_{k=1}^{r}\mathrm{sd}^M((f_k,M')\times \nu^{(M')}_{\triangle^m}\circ \mathrm{sd}^{M'}(\pi_2)))\circ \mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ \mathrm{sd}^{N_f}(\pi_2))) \tag{2}$$

where $M = \max\{N_{f_{ij}}\}_{1\leq i\leq r, 1\leq j\leq r_i}$ and $M' = \max\{N_{f_i}\}_{1\leq i\leq r}$. Now let $M_i = \max\{N_{f_{i,j}}\}_{0\leq j\leq n_i}$ and recall that

$$(f,M) = (f,M_i)\circ \nu^{(M-M_i)}_{\mathrm{sd}^M(X\times\triangle^m)}$$

We may deduce from this that Expression (2) can be written

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij},M_i)\circ \nu^{(M-M_i)}_{\mathrm{sd}^{M_i}(X\times\triangle^m)})\circ(\bigvee_{k=1}^{r}\mathrm{sd}^M((f_k,M')\times \nu^{(M')}_{\triangle^m}\circ \mathrm{sd}^{M'}(\pi_2)))$$
$$\circ\, \mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ \mathrm{sd}^{N_f}(\pi_2))).$$

This can be written

$$(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij},M_i))\circ(\bigvee_{k=1}^{r}\nu^{(M-M_k)}_{\mathrm{sd}^M(X^{\vee r_k}\times\triangle^m)}\circ \mathrm{sd}^M((f_k,M')\times \nu^{(M')}_{\triangle^m}\circ \mathrm{sd}^{M'}(\pi_2)))$$
$$\circ\, \mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\triangle^m}\circ \mathrm{sd}^{N_f}(\pi_2))).$$

Using the commutativity of Diagram 1 we see that this is equal to

$$\Big(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij},M_i)\Big)\circ\Big(\bigvee_{k=1}^{r}\mathrm{sd}^{M_k}((f_k,M')\times\nu_{\triangle^m}^{(M')}\circ\mathrm{sd}^{M'}(\pi_2))\circ\nu^{M-M_k}_{\mathrm{sd}^{M'+M_k}(X\times\triangle^m)}\Big)$$
$$\circ\,\mathrm{sd}^{M+M'}((f,N_f)\times(\nu_{\triangle^m}^{(N_f)}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

Once again using Diagram 1, we can rewrite this as

$$\Big(\bigvee_{i=1}^{r}\Big(\bigvee_{j=1}^{r_i}(f_{ij},M_i)\Big)\circ\mathrm{sd}^{M_i}((f_i,M_{f_i})\times(\nu_{\triangle^m}^{(M_{f_i})}\circ\mathrm{sd}^{M_{f_i}}(\pi_2)))\circ\nu^{M+M'-M_i-M_{f_i}}_{\mathrm{sd}^{M'+M_i}(X\times\triangle^m)}\Big)$$
$$\circ\,\mathrm{sd}^{M+M'}((f,N_f)\times(\nu_{\triangle^m}^{(N_f)}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

The above expression is equal to

$$\bigvee_{i=1}^{r}(\overline{\gamma(f_i,f_{i1},\dots,f_{ir_i})},M+M')\circ\mathrm{sd}^{M+M'}((f,N_f)\times(\nu^{(N_f)}_{\mathrm{sd}^{M+M'}(\triangle^m)}\circ\mathrm{sd}^{N_f}(\pi_2))).$$

By our argument on the last page, this is equal to

$$\gamma(f,\gamma(f_1,f_{1,1},\dots f_{1,n_1})\dots,\gamma(f_1,f_{r,1},\dots f_{r,n_r}))$$

as desired.

The identity element of the operad is $\mu_X : X \to \mathrm{Ex}^\infty(X)$. Verifying the equivariance axioms is straightforward, it is almost exactly the same as verifying them for the topological coendomorphism operad. Therefore we have defined an operad. □

It remains only to define simplicial coalgebras, which proceeds exactly as one would expect.

Definition 3.14. Let $\mathscr{P}$ be an operad in simplicial sets. We shall say that a finite simplicial set X is a $\mathscr{P}$*-coalgebra* if there exists an operadic morphism $\Phi : \mathscr{P} \to \mathsf{CoEnd}_{\mathrm{Set}_\triangle}(X)$.

Lastly we define E_n-algebras in $\mathrm{Set}_\triangle$. The W-construction of an operad in simplicial sets is defined in [2].

Definition 3.15. In simplicial sets, an E_n*-coalgebra* is a coalgebra over the W-construction of the Barratt-Eccles E_n-operad.

3.2. **Simplicial suspensions are E_n-coalgebras.** In this section, in direct analogy with [6, Theorem 2.22] in topological spaces, we aim to show that simplicial suspensions are E_n-coalgebras. The strategy of this proof is as follows. First we transfer the little n-discs operad $\mathbb{D}_n$, the topological coendomorphism operad and the operad morphism Φ between them into the category of simplicial sets using the simplicial chains functor $\mathsf{Sing}_\bullet$. We then use the homotopy transfer principle to lift this to a morphism from a cofibrant replacement of $\mathbb{D}_n$ to the simplicial coendomorphism operad.

The precise statement of the simplicial version of Theorem 2.4 is as follows.

Theorem 3.16. *Let $n \in \mathbb{N}$ and $\Sigma^n X$ be the n–fold suspension of a finite pointed simplicial set X. Then $\Sigma^n X$ has the structure of an E_n-coalgebra.*

Our proof of this theorem requires that the Cartesian product commutes with the geometric realization functor. This is actually not true in general. Therefore, we shall need to restrict from the category of all topological spaces to the category of compactly generated Hausdorff spaces and we take our product to be the Kelley product.

We also wish to be able to transfer operads from topological space to simplicial sets. This is made possible by the following definition.

Definition 3.17. Let $\mathscr{P}$ be an operad in Top. We define an operad $\mathsf{Sing}_\bullet\mathscr{P}$ over $\mathrm{Set}_\triangle$ with arity n component

$$(\mathsf{Sing}_\bullet\mathscr{P})(n) := \mathsf{Sing}_\bullet(\mathscr{P}(n))$$

where $\mathsf{Sing}_\bullet$ is the singular chains functor. The action of $\sigma \in \mathbb{S}_n$ on $\mathsf{Sing}_\bullet\mathscr{P}(n)$ is given by $\mathsf{Sing}_\bullet\mathscr{P}(n) * \sigma := \mathsf{Sing}_\bullet(\mathscr{P}(n) * \sigma)$. The operadic composition map is $\gamma_{\mathsf{Sing}_\bullet\mathscr{P}} := \mathsf{Sing}_\bullet(\gamma_{\mathscr{P}})$ and we take the unit to be the simplex $[\triangle^0 \to 1_{\mathsf{Top}}] \in \mathsf{Sing}_\bullet\mathscr{P}(1)$.

Remark 3.18. The operad composition map in the definition above is well-defined because $\mathsf{Sing}_\bullet$ is right adjoint to the geometric realization. This implies that it preserves limits, and in particular, products.

We can actually define $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ to be an alternative coendomorphism operad. The following theorem gives us a precise description of it.

Lemma 3.19. *Let X be a simplicial set with only finitely many nondegenerate simplices. The operad $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ is isomorphic to the simplicial operad $Q(X)$ with arity r component equal to*

$$Q(X)(r) := \mathrm{Set}_\triangle(X, \mathsf{Sing}_\bullet|X^{\vee r}|).$$

Let $f \in Q(X)(r)_m$ and let $f_i \in Q(X)(n_i)_m$ for $1 \le i \le r$. The operadic composition map

$$\gamma : Q(X)(r) \times Q(X)(n_1) \times \cdots \times Q(X)(n_r) \to Q(X)(n_1 + \cdots + n_r)$$

is given by the adjoint under the Top-Set$_\triangle$ *adjunction of $F : |X \times \triangle^m| \to |X^{\vee n_1 + \cdots + n_r}|$, where F is defined by*

$$|X \times \triangle^m| \xrightarrow{|\mathrm{id} \times \delta_{\triangle^m}|} |X \times \triangle^m \times \triangle^m| \xrightarrow{a} |X \times \triangle^m| \times |\triangle^m| \xrightarrow{|f| \times \mathrm{id}} |\mathsf{Sing}_\bullet|X^{\vee r}|| \times |\triangle^m| \xrightarrow{\epsilon_{X^{\vee r}} \times \mathrm{id}}$$
$$|X^{\vee r}| \times |\triangle^m| \xrightarrow{b} |X \times \triangle^m|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |f_i|} \bigvee_{i=1}^r |\mathsf{Sing}_\bullet|X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r \epsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}| \xrightarrow{c} |X^{\vee n_1 + \cdots + n_r}|$$

where

- *$\delta_{\triangle^m} : \triangle^m \to \triangle^m \times \triangle^m$ is the diagonal map.*
- *for Y a topological space, the map $\epsilon_Y : |\mathsf{Sing}_\bullet(Y)| \to Y$ is the counit of the adjunction between topological spaces and simplicial sets.*
- *$a : |X \times \triangle^m \times \triangle^m| \to |X \times \triangle^m| \times |\triangle^m|$ is an isomorphism, as $\times$ commutes with geometric realisation.*
- *$b : |X^{\vee r}| \times |\triangle^m| \to |X \times \triangle^m|^{\vee r}$ is an isomorphism, as both $\times$ and the wedge sum commute with geometric realisation.*
- *$c : \bigvee_{i=1}^r |X^{\vee n_i}| \to |X^{\vee n_1 + \cdots + n_r}|$ is an isomorphism, as the wedge sum commutes with geometric realisation.*

For each $\sigma \in \mathbb{S}_r$, there is a map $\sigma^* : X^{\vee r} \to X^{\vee r}$ given by permuting the terms of the wedge sum by σ. The symmetric structure on $Q(X)(r)$ is defined by post-composition with the morphism $\mathsf{Sing}_\bullet|\sigma^*|$.

Proof. We can write

$$\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))(r) = \mathsf{Sing}_\bullet \mathrm{Map}_{\mathsf{Top}}(|X|, |X^{\vee r}|) \cong \mathrm{Set}_\triangle(X, \mathsf{Sing}_\bullet|X^{\vee k}|).$$

because, for all $K \in \mathrm{Set}_\triangle$ and $Y \in \mathsf{Top}$, we have

$$\mathsf{Top}(|\triangle^m|, \mathrm{Map}_{\mathsf{Top}}(|K|, Y)) \cong \mathsf{Top}(|\triangle^m| \times |K|, Y)$$

by tensor-hom adjunction. Here it is critical to distinguish between the simplicial mapping space and the hom-set. We then have

$$\mathsf{Top}(|\triangle^m| \times |K|, Y) \cong \mathsf{Top}(|\triangle^m \times K|, Y)$$

by the identity $|X| \times |Y| \cong |X \times Y|$ and finally we have

$$\mathsf{Top}(|\triangle^m \times K|, Y) \cong \mathrm{Set}_\triangle(\triangle^m \times K, \mathsf{Sing}_\bullet Y)$$

by adjunction.

Secondly, it remains to check that operad morphisms are as described in the statement of the lemma. We can describe the induced operad structure on $\mathsf{Top}(|\triangle^m \times X|, |X^{\vee r|})$ quite easily. For $f \in \mathsf{Top}(|\triangle^m \times X|, |X^{\vee r|})$ and $f_i \in \mathsf{Top}(|\triangle^m \times X|, |X^{\vee n_i|})$ the composite $\gamma(f, f_1, \ldots f_n)$ is the function

$$F : |X \times \triangle^m| \xrightarrow{|\mathrm{id} \times \delta_{\triangle^m}|} |X \times \triangle^m \times \triangle^m| \xrightarrow{a} |X \times \triangle^m| \times |\triangle^m| \xrightarrow{f \times \mathrm{id}}$$
$$|X^{\vee r}| \times |\triangle^m| \xrightarrow{b} |X \times \triangle^m|^{\vee r} \xrightarrow{\bigvee_{i=1}^r f_i} \bigvee_{i=1}^r |X^{\vee n_i}| \xrightarrow{c} |X^{\vee n_1 + \cdots + n_r}|$$

The isomorphism

$$G : \mathrm{Set}_\triangle(\triangle^m \times X, \mathsf{Sing}_\bullet|X^{\vee r}|) \xrightarrow{\sim} \mathsf{Top}(|\triangle^m \times X|, |X^{\vee r}|)$$

can be written by

$$f \mapsto \epsilon_{X^{\vee r}} \circ |f|.$$

Therefore the composition map is exactly as described. □

The simplicial coendomorphism operad and the operad $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ are equivalent.

Theorem 3.20. *Let X be a finite simplicial set. Then the simplicial coendomorphism operad and the operad* $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ *are weakly equivalent.*

It follows from this that the simplicial coendomorphism operad is isomorphic to the topological coendomorphism operad in common homotopy category of topological spaces and simplicial sets.

We shall prove this by constructing a zig-zig involving a third operad, which we define will first.

Definition 3.21. Let X be a finite simplicial set. Then the *mixed coendomorphism operad* $R(X)$ has arity r component

$$R(X)(r) = \mathrm{Set}_\triangle(X, \mathrm{Ex}^\infty(\mathsf{Sing}_\bullet|X^{\vee r}|)).$$

For each $\sigma \in \mathbb{S}_r$, there is a map $\sigma^*: X^\vee \to X^\vee$ given by permuting the terms of the wedge sum by σ. The symmetric structure on $R(X)(r)$ is defined by post-composition with the morphism $\mathrm{Ex}^\infty(\mathsf{Sing}_\bullet|\sigma^*|)$. We shall define the operadic composition map using both the sd–Ex and the simplicial chains-geometric realization adjunctions consecutively. Let $f \in Q(X)(r)_m$ and $f_i \in Q(X)(n_i)_m$ for $1 \le i \le r$, then the operadic composition map

$$\gamma : R(X)(r) \times R(X)(n_1) \times \cdots \times R(X)(n_r) \to R(X)(n_1 + \cdots + n_r)$$

is defined to be $\overline{F}$ which is adjoint, under the sd-Ex adjunction, of the morphism, $F : \mathrm{sd}^{N_f}(X \times \triangle^m) \to \mathsf{Sing}_\bullet|X^{\vee n_1+\cdots+n_r}|$. F is itself an adjoint, this time under the geometric realization –simplicial chains adjunction, of a morphism $G : |\mathrm{sd}^{N+N_f}(X \times \triangle^m)| \to |X^{\vee n_1+\cdots+n_r}|$ which we define to be the composite

$$\begin{aligned}
&|\mathrm{sd}^{N+N_f}(X \times \triangle^m)| \xrightarrow{|\mathrm{sd}^N(\delta_{\triangle^m})|} |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m))| \\
&\xrightarrow{|\mathrm{sd}^N(\mathrm{id} \times \mathrm{sd}^{N_f}(\pi_2))|} |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(\triangle^m))| \xrightarrow{a} |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)| \\
&\xrightarrow{\mathrm{sd}^N((f,N_f)\times \mathrm{id})} |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)| \xrightarrow{b} |\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m| \xrightarrow{c} |X^{\vee r} \times \triangle^m| \\
&\xrightarrow{d} |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \xrightarrow{e} |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |f_i|} \bigvee_{i=1}^r |\mathsf{Sing}_\bullet|X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r \epsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}|
\end{aligned}$$

where

- N is the integer $\max(N_{f_1}, \ldots, N_{f_n})$.
- and for Y a topological space, the map $\epsilon_Y : |\mathsf{Sing}_\bullet(X^{\vee r})| \to Y$ is the counit of the adjunction between topological spaces and simplicial sets.
- $\delta_{\mathrm{sd}^{N_f}(X\times\triangle^m)} : \mathrm{sd}^{N_f}(X \times \triangle^m) \to \mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m)$ is the diagonal map.
- $\pi_2 : X \times \triangle^m \to \triangle^m$ is the projection.
- $a : |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(\triangle^m))| \to |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)|$ is the map $|\mathrm{sd}^N(\mathrm{id} \times \nu_{\triangle^m} \circ \cdots \circ \nu_{\mathrm{sd}^{N_f - 1}\triangle^m})|$.
- $b : |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)| \to |\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m|$ is a homeomorphism, by Lemma 2.6, which states that there is a homeomorphism $h_Z : |\mathrm{sd}(Z)| \to |Z|$ for every simplicial set Z (although this homeomorphism is not necessarily natural for simplicial morphisms $Z \to Z'$).
- $c : |\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m| \xrightarrow{c} |X^{\vee r} \times \triangle^m|$ is the composite

  $$|\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m| \xrightarrow{p} |\mathsf{Sing}_\bullet|X^{\vee r}|| \times |\triangle^m| \xrightarrow{|\epsilon_{X^{\vee r}}| \times \mathrm{id}} |X^{\vee r}| \times |\triangle^m| \xrightarrow{q} |X^{\vee r} \times \triangle^m|$$

  where p and q are isomorphisms as the Kelley product commutes with geometric realisation.
- $d : |X^{\vee r} \times \triangle^m| \to |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)|$ is the homeomorphism that exists by Lemma 2.6.
- $e : |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \to |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)|^{\vee r}$ is a homeomorphism because wedge sum commutes with geometric realization.
- $f : \bigvee_{i=1}^r |X^{\vee n_i}| \to |X^{\vee n_1+\cdots+n_r}|$ is a homeomorphism, as the wedge sum commutes with geometric realisation.

We now start the proof of Theorem 3.20.

Proof of Theorem 3.20. Since, by Lemma 3.19, the operad $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}}(|X|))$ is isomorphic to $Q(X)(r)$, it suffices to construct a zig-zag of weak equivalences

$$\mathsf{CoEnd}(X) \xrightarrow{p} R(X) \xleftarrow{q} Q(X).$$

We define $p(r)$ to be the morphism

$$\mathrm{Set}_\triangle(X, \mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})) : \mathrm{Set}_\triangle(X, \mathrm{Ex}^\infty(X^{\vee r})) \to \mathrm{Set}_\triangle(X, \mathrm{Ex}^\infty(\mathsf{Sing}_\bullet|X^{\vee r}|))$$

where $\upsilon_{X^{\vee r}} : X^{\vee r} \to \mathsf{Sing}_\bullet|X^{\vee r}|$ is the unit of the singular chains – geometric realization adjunction. Observe that $\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}}) : \mathrm{Ex}^\infty(X^{\vee r}) \to \mathrm{Ex}^\infty(\mathsf{Sing}_\bullet|X^{\vee r}|)$ is a weak equivalence between fibrant simplicial sets. Hence it is a homotopy equivalence, and the functor $\mathrm{Set}_\triangle(X,-)$ preserves homotopy equivalences. Hence p is a weak equivalence.

It remains to check that it induces a morphism of operads. We check this directly. Note first that $\mathrm{Set}_\triangle(X, \mathrm{Ex}^\infty(\upsilon_{X^{\vee r}}))(f) = \mathrm{Ex}^\infty(\upsilon_{X^{\vee r}}) \circ f$. Then observe that $N_{\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\circ f} = N_f$ and that we have $\max(N_{\upsilon_{\mathrm{Ex}^\infty(X^{\vee r})}\circ f_1}, \dots, N_{\upsilon_{\mathrm{Ex}^\infty(X^{\vee r})}\circ f_n}) = \max(N_{f_1}, \dots, N_{f_n})$. Then observe that the morphism

$$|\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)| \xrightarrow{\mathrm{sd}^N((\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\circ f, N_f)\times \mathrm{id})} |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)|$$

factors as

$$|\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)| \xrightarrow{\mathrm{sd}^N((f,N_f)\times \mathrm{id})} |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \xrightarrow{\mathrm{sd}^N(\upsilon_{X^{\vee r}}\times \mathrm{id})} |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)|$$

Moreover, having first observed that the following diagram is commutative

$$\begin{tikzcd}
|\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \arrow[r, "|\mathrm{sd}^N(\upsilon_{X^{\vee r}}\times \mathrm{id})|"] \arrow[d, "h_{(X^{\vee r}\times\triangle^m)}"] & |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)| \arrow[d, "h_{\mathsf{Sing}_\bullet|X^{\vee r}|\times\triangle^m}"] \\
|(X^{\vee r} \times \triangle^m)| \arrow[r, "|(\upsilon_{X^{\vee r}}\times \mathrm{id})|"] & |(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)|,
\end{tikzcd}$$

where $h_Z : |\mathrm{sd}\, Z| \to |Z|$ is the map that exists by Lemma 2.6, we see that the composite

$$|\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \xrightarrow{|\mathrm{sd}^N(\upsilon_{X^{\vee r}}\times \mathrm{id})} |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)| \xrightarrow{b} |\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m|$$
$$\xrightarrow{c} |\mathsf{Sing}_\bullet|X^{\vee r}|| \times |\triangle^m| \xrightarrow{|\epsilon_{X^{\vee r}}|\times \mathrm{id}} |X^{\vee r}| \times |\triangle^m| \xrightarrow{d} |X^{\vee r} \times \triangle^m| \xrightarrow{e} |\mathrm{sd}^N(X \times \triangle^m)|^{\vee r}$$

is an isomorphism by the *triangle identities* for the $\mathsf{Sing}_\bullet$–$|-|$ adjunction. Explicitly, the (left) triangle identity for an adjunction $L \dashv R$ with unit $\eta : id_X \to R \circ L$ and counit $\epsilon : L \circ R \to id_Y$ states that the natural transformation of functors defined as the composite

$$L \xrightarrow{L\eta} LRL \xrightarrow{\epsilon L} L$$

is the identity transformation. Upon further observing that, for the same reason, the composite

$$|\mathrm{sd}^N(X \times \triangle^m)|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\circ f_i|} \bigvee_{i=1}^r |\mathsf{Sing}_\bullet|X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r \epsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}|$$

is exactly the map

$$|\mathrm{sd}^N(X \times \triangle^m)|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |f_i|} \bigvee_{i=1}^r |X^{\vee n_i}|,$$

it becomes obvious that γ commutes with p, and so p is a weak equivalence of operads.

Similarly, we define $q(r)$ to be the morphism

$$\mathrm{Set}_\triangle(X, \mu_{\mathsf{Sing}_\bullet|X^{\vee r}|}) : \mathrm{Set}_\triangle(X, \mathsf{Sing}_\bullet|X^{\vee r}|) \to \mathrm{Set}_\triangle(X, \mathrm{Ex}^\infty(\mathsf{Sing}_\bullet|X^{\vee r}|)).$$

This is a weak equivalence of simplicial sets for exactly the same reasons that $p(r)$ is. Observe that $N_{q(r)(f)} = 0$ for all $f \in Q(X)(r)$. It follows from the form of the operad maps that the morphism q identifies $Q(X)$ with a suboperad of $R(X)(r)$. In particular, q is a morphism of operads, and so a weak equivalence of operads. □

Finally, we can prove the main result of this section.

Theorem 3.16. Let $\Sigma^n X$ be the n–fold suspension of a simplicial set X. As $|\Sigma X|$ is a CW-complex, it is in Top. Suspensions are a particular kind of finite colimits, and the geometric realization functor commutes with all colimits as it is a right adjoint, so suspensions commute with geometric realization and thus that $|\Sigma^n X|$ is a coalgebra over the little n–discs operad in Top. This coalgebra structure is an operadic morphism $\Phi : \mathbb{D}_n \to \mathrm{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|)$. As discussed above, we can use $\mathsf{Sing}_\bullet$ to transfer these operads and this algebra structure to the category of simplicial sets, producing the following morphism of operads

$$\mathsf{Sing}_\bullet(\Phi) : \mathsf{Sing}_\bullet(\mathbb{D}_n) \to \mathsf{Sing}_\bullet(\mathrm{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|))$$

Theorem 3.20 tells us that there is a weak equivalence between $\mathrm{CoEnd}_{\mathsf{Set}_\triangle(X)}$ and $\mathsf{Sing}_\bullet(\mathrm{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|))$. Observe that in each arity $\mathrm{CoEnd}_{\mathsf{Set}_\triangle}(X)(n)$ is a mapping space where the target is a Kan complex, hence Kan itself and a fibrant operad in the operadic model structure. By its construction, in each arity $\mathsf{Sing}_\bullet \mathrm{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|)$ is a singular complex and thus as an operad it is also fibrant.

Since we have a weak equivalence between fibrant operads, over the cofibrant replacement $(\mathsf{Sing}_\bullet \mathbb{D}_n)_\infty$ of $\mathsf{Sing}_\bullet \mathbb{D}_n$ we have an induced bijection between the homotopy classes of morphisms of operads

$$[(\mathsf{Sing}_\bullet \mathbb{D}_n)_\infty, \mathrm{CoEnd}_{\mathsf{Set}_\triangle}(\Sigma^n X)] \cong [(\mathsf{Sing}_\bullet \mathbb{D}_n)_\infty, \mathsf{Sing}_\bullet \mathrm{CoEnd}_{\mathsf{Top}}(|\Sigma^n X|)].$$

So we can choose a morphism $\phi : (\mathsf{Sing}_\bullet \mathbb{D}_n)_\infty \to \mathrm{CoEnd}_{\mathsf{Set}_\triangle}(\Sigma^n X)$, such that ϕ is homotopy equivalent to $\mathsf{Sing}_\bullet \Phi$.

Finally to prove that n-fold suspensions are E_n-algebras it suffices to note that all topological operads are fibrant and so the weak equivalence between the little n-discs operad and the geometric realization of the Barratt-Eccles E_n-operad remains one when taking the $\mathsf{Sing}_\bullet$ functor. The Barratt-Eccles E_n-operad $\Gamma^{(n)}$ is weakly equivalent to $\mathsf{Sing}_\bullet|\Gamma^{(n)}|$, and in particular, $(\mathsf{Sing}_\bullet \mathbb{D}_n)_\infty$ can be taken to be the Boardman-Vogt resolution of $\Gamma^{(n)}$; the operad $W(\triangle^1, \Gamma^{(n)})$. $\square$

REFERENCES

[1] Clemens Berger. Combinatorial models for real configuration spaces and E_n-operads. *Contemp. Math.*, ***202***, 1997.

[2] Clemens Berger and Ieke Moerdijk. The Boardman-Vogt resolution of operads in monoidal model categories. *Topology*, 45(5):807–849, 2006.

[3] Frederick R. Cohen, Thomas J. Lada, and J. P. May. *The homology of iterated loop spaces*. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[4] Edward B. Curtis. Simplicial homotopy theory. *Advances in Math.*, ***6***, 1971.

[5] Eldon Dyer and R. K. Lashof. Homology of iterated loop spaces. *Amer. J. Math. 84 , 35–88.*, 1962.

[6] Oisín Flynn-Connolly, José M. Moreno-Fernández, and Felix Wierstra. A recognition principle for iterated suspensions as coalgebras over the little cubes operad. *arXiv:2210.00839*, 2022.

[7] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Birkhäuser Verlag, Basel, 1999.

[8] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 346. Springer, Heidelberg, 2012.

[9] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin-New York, 1972.

[10] José M. Moreno-Fernández and Felix Wierstra. Iterated suspensions are coalgebras over the little disks operad. *arXiv:1909.11043*, 2019.

LEIDEN UNIVERSITY, LEIDEN, THE NETHERLANDS
Email address: oisinflynnconnolly@gmail.com