# Integral arithmetic means of the first Fibonacci numbers

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# Outline


1. The arithmetic-mean question
2. Known results and conjectures
3. Conjecture 1: odd integral means
4. Conjecture 2: consecutive pairs
5. What's next

# The arithmetic-mean question

# Our object of study

Let $F_n$ be the Fibonacci numbers, with the convention $F_1 = F_2 = 1$. In 2022, Fatehizadeh and Yaqubi [1] introduced the *arithmetic mean of the first n Fibonacci numbers*:

$$A_F(n) \coloneqq \frac{1}{n} \sum_{i=1}^{n} F_i.$$

**Question**

*When is $A_F(n)$ an integer?*

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The key tool is the classical identity $\sum_{i=1}^{n} F_i = F_{n+2} - 1$

$$A_F(n) = \frac{F_{n+2} - 1}{n} \qquad \Longrightarrow \qquad A_F(n) \in \mathbb{Z} \iff n \mid F_{n+2} - 1.$$

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At present there appear to be four papers studying this problem [1, 3, 2, 4], appearing in 2022, 2024, 2025, 2025.

# Known results and conjectures

# Fatehizadeh and Yaqubi's results

They had the following results:

1. $A_F(2 \cdot 3^n)$ is an integer when $n \geq 3$.
2. $A_F(2^n \cdot 3^m \cdot 5^l)$ is an integer when $n \geq 3$ and $m \geq 1$. Moreover, it is divisible by 18.
3. $A_F(p)$ is never an integer for $p$ an odd prime[1].

They conjecture, with compelling numerical evidence, the following:

1. There are infinitely many odd integers $n$ such that $A_F(n)$ is an integer. [A331976]
2. There are infinitely many pairs of integers $(n, n+1)$ such that $(A_F(n), A_F(n+1))$ are integers. [A111035.]

[1] They case-check, my paper has a direct proof using Binet that computes the two possible remainders

# Křížek and Somer's results

This is a long (41 pages) and authoritative structural account on the topic.
They generalise the results from Fibonacci numbers to second-order sequences of the form.

$$W_{n+2} = aW_{n+1} + bW_n \qquad A_W(n) = \frac{1}{n}\sum_{i=1}^{n} W_i$$

**Question**

*When is $A_W(n)$ an integer?*

This allows them to give a structural account of FY's results in terms of the discriminant of $W$.

# Conjecture 1: odd integral means

# Fatehizadeh and Yaqubi Conjecture 1

**There are infinitely many odd integers $n$ such that $A_F(n)$ is an integer.**

1. To prove the result, one just needs a construction.
2. Easier conjecture: there are infinitely many $n$ such that $n \mid F_n$.
3. Observation: $n \mid k \implies F_n \mid F_k$. In particular: $n \mid F_n \implies F_n \mid F_{F_n}$.
4. Let $m = F_n$. When is $A_F(F_n)$ an integer? Can use Python code to search for $n$ such that $F_n \mid F_{F_n+2} - 1$.
5. Lots of hits:

$$\{1, 2, 3, 12, 24, 34, 36, 46, 48, 60, 68, 72, 92, 94, 96, 106, 108, 120, 144, \cdots\}$$

   I call such $n$ the *self-summable Fibonacci numbers* (A383021)
6. We have that $F_n$ is odd iff $n \equiv 1, 2 \mod 3$. Leaves us with

$$\{1, 2, 34, 46, 68, 92, 94, 106, 166, 188, 212, 214, 226, 274, 332, 334, 346\}$$

7. Most (not all!) of these are of the form $2p$ or $4p$ where $p \equiv 2 \mod 3$ and $p \equiv \pm 2 \mod 5$.

# Proof of conjecture

**Theorem**

*Suppose that* $n = 2p$ *or* $n = 4p$*, where* $p$ *is an odd prime such that* $p \equiv 2 \pmod 3$ *and* $p \equiv \pm 2 \pmod 5$*. Then* $F_n \equiv 1 \pmod 2$ *and* $F_{F_n+2} - 1 \equiv 0 \pmod{F_n}$.

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**Idea**: reduce the index twice, using Pisano periods $\pi(\cdot)$ and Wall's theorems about them.

1. $F_{F_{2p}+2} \bmod F_{2p}$ depends only on the index mod $\pi(F_{2p})$, and $\pi(F_n) \mid 2n$ for even $n$ (easy to show), so $\pi(F_{2p}) \mid 4p$.

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2. So it suffices to know $F_{2p}$ mod $4p$, hence $2p$ mod $\pi(4p)$, where

$$\pi(4p) = \operatorname{lcm}\big(\pi(4), \pi(p)\big) = \operatorname{lcm}\big(6, \pi(p)\big).$$

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Hence $F_{2p} + 2 \equiv 1$, and

$$F_{F_{2p}+2} - 1 \equiv F_{(F_{-2}+2) \bmod 4p} - 1 = F_1 - 1 \equiv 0 \pmod{F_{2p}}. \qquad \square$$

# Solution to Conjecture 1

**Theorem (FC.)**

*Suppose that $n = 2p$ or $n = 4p$, where $p$ is an odd prime such that $p \equiv 2 \pmod 3$ and $p \equiv \pm 2 \pmod 5$. Then $F_n \equiv 1 \pmod 2$ and $F_{F_n+2} - 1 \equiv 0 \pmod{F_n}$.*

The congruences $p \equiv 2 \pmod 3$ and $p \equiv \pm 2 \pmod 5$ are simultaneously satisfiable, so by Dirichlet's theorem there are infinitely many primes $p$ satisfying them. So the conclusion follows.

**Corollary (FC.)**

*There are infinitely many odd integers $n$ such that $A_F(n)$ is an integer.*

# Conjecture 2: consecutive pairs

# Fatehizadeh and Yaqubi Conjecture 2

**There are infinitely many pairs of integers $(n, n+1)$ such that $(A_F(n), A_F(n+1))$ are integers.**

1. This was proven by Luca and Marques. A rather ingenious construction using recursion and the number 108.
2. They set up a sequence $\{k_i\}_{i\in\mathbb{N}}$ satisfying some properties and then construct $n$ as a ratio between $F_{108k_i}/F_{k_i}$.
3. The proof makes use of $p$-adic valuations and the identity $F_a - F_b = F_{\frac{a-b}{2}} L_{\frac{a+b}{2}}$ when $a \equiv b \pmod 4$

# What's next

# Odd self-summable Fibonacci pairs

Among the self-summable $k$ (searching $< 12000$), odd $F_k$ sometimes has an AF-integral *neighbour*, giving us a consecutive pair.

| $k = 2p$ | $p$ |
|---|---|
| 94 | 47 |
| 214 | 107 |
| 334 | 167 |
| 1294 | 647 |
| 2734 | 1367 |
| 4414 | 2207 |
| 4894 | 2447 |
| 11014 | 5507 |

**Right pairs** $(F_k, F_k + 1)$. These all seem to be $k = 2p$ with $p \equiv 47 \pmod{60}$, and appear rather common.

**Left pairs** $(F_k - 1, F_k)$, These seem far rarer. The only one I have found is

$$k = 41474 = 2 \cdot 89 \cdot 233,$$

with $k \equiv 2 \pmod{12}$. $F_k$ has 8668 digits!

# Further questions

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2. Are there infinitely many *right pairs* $(F_{2p}, F_{2p}+1)$ with $p \equiv 47 \pmod{60}$? Infinitely many *left pairs*, or even a second one?
3. Is there an approach to Conjecture 2 via self-summable Fibonacci numbers?
4. Provide a classification of all (or most) self-summable Fibonacci numbers in terms of their prime factorization. Are there other infinite families?
5. Extend to other number sequences.

A. Fatehizadeh and D. Yaqubi.
Average of the Fibonacci numbers.
*J. Integer Seq.*, 25(2):Art. 22.2.6, 10, 2022.

O. Flynn-Connolly.
On the divisibility of sums of Fibonacci numbers.
*Integers*, 25:Paper No. A87, 8, 2025.

M. Křížek and L. Somer.
Period lengths modulo $n$ and average of terms of second order linear recurrences.
*Integers*, 24:Paper No. A36, 41, 2024.

Florian Luca and Diego Marques.
On integral consecutive arithmetic means of the first Fibonacci numbers.
*Arch. Math. (Basel)*, 125(5):505–511, 2025.