Final coalgebras in R-modules

Dutch Categories And Types Seminar

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Background

Final coalgebras

1. A **coalgebra** for an endofunctor $F\colon \mathscr{C} \to \mathscr{C}$ is a pair $(X, f\colon X \to FX)$.
2. The F-coalgebras form a category with morphisms $u\colon (X, f) \to (Y, g)$

$$\begin{array}{ccc} X & \xrightarrow{f} & FX \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle Fu} \\ Y & \xrightarrow{g} & FY \end{array}$$

3. The terminal object $(\nu F, t)$ in the category of coalgebra is called the **final coalgebra**. It contains all possible behaviours of the system: the terminal map $X \to \nu F$, sends an object to its set of all possible behaviors.
4. The map $t\colon \nu F \to F(\nu F)$ is an isomorphism. (Lambek's lemma)

Final coalgebras via transfinite induction in Set

Consider the terminal map $m\colon F1 \to 1$.

$$\cdots \xrightarrow{F^3 m} F^3 1 \xrightarrow{F^2 m} F^2 1 \xrightarrow{Fm} F1 \xrightarrow{m} 1$$

Let the limit of the sequence above be $(V_\omega, (\ell_i)_{i\geq 0})$. We have a map $m_\omega\colon FV_\omega \to V_\omega$ induced by the diagram $FV_\omega \xrightarrow{F\ell_n} F^{n+1}1$. **This is always a mono but NOT always an iso (P_{fin}).** So you iterate again obtaining a diagram of monos

$$\cdots \xrightarrow{F^3 m} F^3 V_\omega \xrightarrow{F^2 m_\omega} F^2 V_\omega \xrightarrow{Fm_\omega} FV_\omega \xrightarrow{m_\omega} V_\omega$$

Take the limit of the above (i.e., the intersection).

Theorem (Worrell, 2005)

*The final coalgebra sequence of a finitary functor $F\colon$ **Set** $\to$ **Set** converges in $\omega + \omega$ steps.*

Final coalgebras via transfinite induction in lfp categories

$$\text{Final coalgebra} = \underbrace{\text{construction}}_{\omega\text{-steps}} + \underbrace{\text{pruning}}_{\omega\text{-steps}}$$

Definition

A **DCC-category** is a locally finitely presentable category where every descending chain of monos or strong epis of a locally presentable object eventually stabilises.

Final coalgebras via transfinite induction in lfp categories

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Definition

A **DCC-category** is a locally finitely presentable category where every descending chain of monos or strong epis of a locally presentable object eventually stabilises. Think: generated by Artinian objects

Examples include sets, posets, vector spaces.

Non-examples include abelian groups: $\mathbb{Z} \xrightarrow{2\cdot} 2\mathbb{Z} \xrightarrow{2\cdot} 4\mathbb{Z} \xrightarrow{2\cdot} 8\mathbb{Z} \cdots$

Theorem (Ademek-Milius-Moss, 2023)

Let $\mathscr{C}$ be a DCC-category. The final coalgebra sequence of a binary intersection preserving, finitary functor $F\colon \mathscr{C} \to \mathscr{C}$ converges in $\omega + \omega$-steps.

Our work

Our question

Ademek-Milius-Moss ask:

Is there a binary intersection preserving, finitary functor $F : \mathbf{Ab} \to \mathbf{Ab}$ *that do not converge in* $\omega \cdot 2$*-steps?*

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Yes.

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Is there a binary intersection preserving, finitary functor $F : \mathbf{Ab} \to \mathbf{Ab}$ *that do not converge in* $\omega \cdot 2$*-steps?*

Yes. In the rest of this talk, we will construct this example and extract some features from it in order to develop a general theory that applies to modules..

Constructing the counterexample: Intuition

1. The Artinian finitely presented abelian groups are the direct sums of finite torsion groups. Under directed colimits, this generates the category $\mathbf{Tors}(\mathbb{Z})$.
2. The category $\mathbf{Tors}(\mathbb{Z})$ is not complete. The limit of the ω^{op}-chain

$$\cdots \longrightarrow \mathbb{Z}/p^4\mathbb{Z} \longrightarrow \mathbb{Z}/p^3\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

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   is the p-adic numbers $\widehat{\mathbb{Z}_p}$ which is torsion-free.
3. Imagine if we have a functor F_p which had that final coalgebra chain?
4. Then $\mathbb{Q} \otimes_{\mathbb{Z}} V_\omega = \mathbb{Q}$ but $\mathbb{Q} \otimes_{\mathbb{Z}} F^n(1) = 0$. So we could look at the functor $F_p(-) \oplus L(\mathbb{Q} \otimes -)$ where $L \colon \mathbb{Q}\mathrm{Vect} \to \mathbb{Q}\mathrm{Vect}$ is some functor that takes $\omega + \omega$ steps to converge.
5. So the combined functor takes $\omega \cdot 3$ steps to converge.
6. The idea is

$$\text{Final coalgebra} = \underbrace{\text{torsion construction}}_{\omega\text{-steps}} + \underbrace{\text{non-torsion construction}}_{\omega\text{-steps}} + \underbrace{\text{pruning}}_{\omega\text{-steps}}$$

Constructing F_p

Let G be a group and fix some $g \in G$:

$$\bar{\star}\colon G_g \times G_g \to G_g, \quad (g_1, i)\bar{\star}(g_2, j) = \begin{cases} (g_1 \star g_2, i+j) & \text{if} \quad i+j < p \\ (g_1 \star g_2 \star g, i+j \bmod p) & \text{if} \quad i+j \geq p \end{cases}$$

1. You can think of the operation here as carrying
2. We have $G_e \cong G \oplus \mathbb{Z}/p\mathbb{Z}$.
3. On any element g in $\mathbb{Z}/p^n\mathbb{Z}$ that is not divisible by p, we obtain $(\mathbb{Z}/p^n\mathbb{Z})_g = \mathbb{Z}/p^{n+1}\mathbb{Z}$.
4. (Context for the experts) This is a **group extension.** We have a short exact sequence

$$0 \to G \to G_g \to \mathbb{Z}/p\mathbb{Z} \to 0$$

Group extension functor

The **group extension functor** $F_p : \mathbf{Ab} \to \mathbf{Ab}$ is defined on objects by

$$F_p(G) = \bigoplus_{g \in G} G_g.$$

On morphisms $f : G \to H$, it is defined by the map $F_p(f) : \bigoplus_{g \in G} G_g \to \bigoplus_{h \in H} H_h$ which maps the summand G_g to $H_{f(g)}$ via the assignment $(a, i) \mapsto (f(a), i)$.

The final coalgebra chain of F_p has p^n-order torsion in $F^n(1)$. So it has the p-adic chain as a subobject in its limit.

So we define $U(-) = F_p(-) \oplus V \otimes_{\mathbb{Q}} (\mathbb{Q} \otimes -)$

Theorem (Basold-FC-Ford)

The functor U is finitary and binary intersection preserving but does not converge in $\omega \cdot 2$ steps.

Some observations on the counterexample

For the counterexample:

- Recall that $\mathbf{Ab} = \mathbf{Mod}(\mathbb{Z})$.
- We critically used the fact that $\mathbf{Tors}(\mathbb{Z}) \subsetneq \mathbf{Ab} = \mathbf{Mod}(\mathbb{Z})$. The category $\mathbf{Tors}(\mathbb{Z})$ is generated by Artinian objects.
- We also have that the (Serre) quotient $\mathbf{Mod}(\mathbb{Z})/\mathbf{Tors}(\mathbb{Z}) \cong \mathbb{Q}\mathbf{Vect}$. This is also DCC!
- The connecting map $UV_\omega \to V_\omega$ is not a monomorphism but it does have a torsion-free kernel.
- In $\mathbf{Mod}(R)$, a map $f\colon M \to N$ is a monomorphism if and only if the kernel $\ker f = \{m \in M\colon f(m) = 0\}$ is equal to $\{0\}$.
- The above two facts hint that we should expand our idea of what it means to be a monomorphism.

Finiteness properties of rings

- A ring is **Noetherian** if every ascending chain of ideals eventually stabilises. Examples: $\mathbb{R}, \mathbb{Z}, \mathbb{Z}[X]$
- It is **Artinian** if every descending chain of ideals eventually stabilises. Example: fields. Non-examples: $\mathbb{Z}$
- An ideal is **prime** if $rs \in I$ iff either $r \in I$ or $s \in I$. In $\mathbb{Z}$, the prime ideals are precisely $p\mathbb{Z}$ where p is prime or zero.
- The **Krull dimension** $\dim R$ of a ring R is the length of its longest ascending chain of **prime** ideals

$$0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq R.$$

Ex: $\dim R = 0$ if R is Artinian, $\dim \mathbb{Z} = 1$ and $\dim R[X] = \dim R + 1$.

Krull filtrations of categories

1. The annihilator of a module $\operatorname{Ann} M = \{r \in R \colon r \cdot m = 0 \text{ for all } m \in M\}$.
2. The **Krull dimension** of a finitely presentable module $\dim M = \dim (R/\operatorname{Ann} M)$ and $\dim 0 = -\infty$. Ex: $\dim \mathbb{Z}/p\mathbb{Z} = 0$.
3. Let R be a Noetherian ring. We filter

$$\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_{\dim R} = \mathbf{Mod}(R) \qquad \text{Example: } \mathbf{Tors}(\mathbb{Z}) \subsetneq \mathbf{Mod}(\mathbb{Z})$$

   where $\mathcal{T}_d$ is the subcategory of $\mathbf{Mod}(R)$ generated under filtered colimits by finitely presentable modules of dimension d or less. In our running example $\mathcal{T}_0 = \mathbf{Tors}(\mathbb{Z})$.
4. There is a coreflection functor $t_d \colon \mathbf{Mod}(R) \to \mathcal{T}_d$. It sends a module M to the union of its finitely presented submodules of dimension less than d. Ie., $t_0(\mathbb{Z}^n \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_n\mathbb{Z}) = \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_n\mathbb{Z}$.
5. The intuition is then that each quotient $\mathcal{T}_{i+1}/\mathcal{T}_i$ is a DCC category.

Factorization systems

Consider:

$$\mathcal{E}_d := \{f : f \text{ is epic and } \ker f \in \mathcal{T}_d\}, \qquad \mathcal{M}_d := \{f : t_d(\ker f) = 0\}.$$

Theorem

The pair $(\mathcal{E}_d, \mathcal{M}_d)$ form orthogonal factorization systems.

This gives us interpolating notions of monomorphism:

$$\{\text{all morphisms}\} = \mathcal{M}_{-1} \supseteq \mathcal{M}_0 \supseteq \mathcal{M}_1 \supseteq \cdots \supseteq \mathcal{M}_n = \{\text{monos}\}$$

One should think of $\mathcal{M}_d$ as the maps that are *monic up to* $\mathcal{T}_d$.

Connecting lemma

The trick is that these factorization systems are related:

Lemma

Let $F\colon \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ be finitary and suppose that it preserves $\mathcal{M}_d$ for $0 \leq d \leq n$. If, for some ordinal $\omega \cdot l$, the connecting maps $m_{\omega \cdot l + k}$ belong to $\mathcal{M}_{d-1}$ for every $k < \omega$, then $m_{\omega \cdot (l+1)}$ belongs to $\mathcal{M}_d$.

(This is essentially an infinite descent argument + some rather annoying commutative algebra arguments).

Convergence in R-modules

Theorem (Basold-FC-Ford)

Let R be a Noetherian ring with Krull dimension n and let $F\colon \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ be a finitary functor that

1. *F preserves countable directed intersections of monomorphisms;*
2. *F preserves $\mathcal{M}_d$ for every $0 \leq d \leq n-1$.*

Then F has a terminal coalgebra obtained in $\omega \cdot (n+2)$ steps.

$$\text{Final coalgebra} = \underbrace{\mathcal{T}_{-1}\text{ construction}}_{\omega\text{-steps}} + \underbrace{\mathcal{T}_0\text{ construction}}_{\omega\text{-steps}} + \cdots + \underbrace{\mathcal{T}_{\dim R}\text{ construction}}_{\omega\text{-steps}} + \underbrace{\text{pruning}}_{\omega\text{-steps}}$$

A closing aside: Bisimulation

Definition

1. A *bisimulation* between F-coalgebras (A, f) and (B, g) is a span of F-coalgebra maps $A \xleftarrow{p_1} S \xrightarrow{p_2} B$ such that the induced map $\langle p_1, p_2 \rangle : S \to A \times B$ is a monomorphism.
2. Two states $x, y \in (A, f)$ are *bisimilar* if there exists a bisimulation $S \to A \times A$ and an $s \in S$ such that $\mathfrak{p}_1(s) = x$ and $p_2(s) = y$.
3. They are behaviourally equivalent if $!_A(x) = !_A(y)$ in νF.

Homological algebra and bisimulation

Bisimulation and Ext

The Ext^1 is the first derived functor of $\hom(-,-)$ Let

$$e := [0 \to \ker\langle Fp_1, Fp_2\rangle \to F\,(A \times_{\nu F} A) \to FA \times_{F(\nu F)} FA \to 0]$$

Theorem (Basold-FC-Ford)

Let $F\colon \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ have a final coalgebra νF. If (A, f) is a coalgebra such that:

1. *the comparison map $\langle Fp_1, Fp_2\rangle\colon F\,(A \times_{\nu F} A) \to FA \times_{F(\nu F)} FA$ is epi;*
2. *$w^*[e] = 0 \in \mathrm{Ext}^1(A \times_{\nu F} A, \ker\langle Fp_1, Fp_2\rangle)$, where $w = \langle fp_1, fp_2\rangle$.*

Then bisimilarity and behavioural equivalence are the same on states of A.

Speculation

Future directions

1. The above theory almost certainly can be extended to general abelian categories with nice filtrations
2. Categories of semimodules over semirings: axiomise this to general localisation of categories.
3. There appears to be a rather close connection between properties of interest in computer science and homological algebra.
4. Good choice of factorization systems is key.
5. It would be particularly interesting and relevant to CS to see how this interacts with homological algebra in characteristic one[1].

[1] A. Connes and C. Consani (2019). "Homological algebra in characteristic one". In: *High. Struct.* 3.1, pp. 155–247. ISSN: 2209-0606.

Supplemetary material

A commutative algebra refresher: modules

- An R-**module** is an abelian group $(M, +)$ equipped with an R-action:

$$R \times M \to M \qquad (r, m) \mapsto rm,$$

  such that: $r(m+n) = rm + rn$, $r(sm) = (r \cdot s)m$, $(r+s)m = rm + sm$ and $1m = m$ for $r, s \in R$ and $m, n \in M$.
- A morphism of R-modules $f : M \to N$ is a morphism of abelian groups that is equivariant with respect to the action of R. We denote the category of R-modules formed thus by $\mathbf{Mod}(R)$.
- For $R = \mathbb{R}$: these are real vector spaces, the action is the scalar multiplication. For $R = \mathbb{R}[x]$: vector spaces with a linear endomorphism etc.
- For $R = \mathbb{Z}$: abelian groups. For $r \in \mathbb{Z}$ the action is $r \cdot m = \underbrace{n + \cdots + n}_{r \text{ times}}$. A module M is *torsion* if for all $m \in M$ there exists $r \in R$ such that $r \cdot m = 0$. It is non-torsion
- There is a tensor product $\otimes_R$ in $\mathbf{Mod}(R)$.

Locally finitely presentable categories

1. A **directed colimit** is a colimit of any diagram $\mathcal{K} \to \mathscr{C}$ where $\mathcal{K}$ is a poset (viewed as a category) where any two elements have an upper bound. Ex: in **Set** these are directed unions of groups. Running assumption: $\mathscr{C}$ has all directed colimits.
2. A functor $F \colon \mathscr{C} \to \mathscr{C}$ is **finitary** if it preserves directed colimits.
3. An object $X \in \mathscr{C}$ is **finitely presentable** if $\mathscr{C}(X, -) \colon \mathscr{C} \to \mathbf{Set}$ preserves filtered colimits. Ex: in **Set** the fp objects are the finite sets. For Noetherian R, in $\mathbf{Mod}(R)$, these are the modules with finitely many generators.
4. A category $\mathscr{C}$ is **locally finitely presentable** (lfp) if it is generated by finitely presentable objects under directed colimits. Ex: **Set** is lfp as every set is the directed union of its finite subsets.