Strictly commutative dg-algebras in positive characteristic

Séminaire d'Homotopie et Géométrie Algébrique
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2. Introduce and motivate strictly commutative dg-algebras in positive characteristic and their basic properties.
3. Obstruction theory in positive characteristic. Define cotriple products, compute the primitive secondary cohomology operations for strictly commutative dg-algebras for and formulate the coherent vanishing of higher Steenrod operations. (Reference: ArXiv: 2404.16681)
4. Introduce an explicit model for the de Rham forms over $\widehat{\mathbb{Z}_p}$. Study what information can be extracted from it. (Reference: Chapter 4 of my thesis)

Part 0: A crash-course in E_∞-algebras

dg-algebras

Definition

A (commutative) dg-algebra is a chain complex (A, d) equipped with a binary (graded commutative) associative multiplication $m : A^p \otimes A^q \to A^{p+q}$ and such that d is a derivation with respect to m. Alternatively it is an algebra over the operad Assoc (or Com) in dg-modules.

Example

Let X be a topological space. Then the cohomology ring $(H^\bullet(X, R), 0)$ equipped with the cup product forms a commutative dg-algebra.

Problem: the cohomology is not a complete invariant of homotopy type.

Example

Let X be a topological space or simplicial set. Then the singular cochains $(C^\bullet(X, R), d)$ equipped with the cochain level cup product forms a dg-algebra that is generally not graded commutative.

E_∞-algebras

Definition

An E_∞-operad is any operadic resolution $\mathcal{E} \xrightarrow{\sim} \mathrm{Com}$ such that the $\mathbb{S}_k$ action on $\mathcal{E}$ is free.

The singular cochain complex $C^\bullet(X, R)$ is an E_∞-algebra. This is a complete homotopy invariant.

Theorem (Mandell, 2003)

Two finite type nilpotent spaces X and Y are weakly equivalent and only if their E_∞-algebras of singular cochains with integral coefficients are quasi-isomorphic as E_∞-algebras.

The Barratt-Eccles operad

Definition

The Barratt-Eccles operad $\mathcal{E}$ is an operad in simplicial sets given in each arity are of the form

$$\mathcal{E}(r)_n = \{(w_0, \dots, w_n) \in \mathbb{S}_r \times \cdots \times \mathbb{S}_r\}$$

equipped with face and degeneracy maps

$$d_i(w_0, \dots, w_n) = (w_0, \dots, w_{i-1}, \hat{w}_i, w_{i+1}, \dots, w_n)$$
$$s_i(w_0, \dots, w_n) = (w_0, \dots, w_{i-1}, w_i, w_i, w_{i+1}, \dots, w_n).$$

$\mathbb{S}_r$ acts on $\mathcal{E}(n)$ diagonally. Finally the compositions are also defined componentwise via the explicit composition law of

$$\gamma : \mathbb{S}(r) \times \mathbb{S}(n_1) \times \cdots \times \mathbb{S}(n_r) \to \mathbb{S}(n_1 + \cdots + n_r)$$
$$(\sigma, \sigma_1, \dots, \sigma_r) \mapsto \sigma_{n_1 \cdots n_r} \circ (\sigma_1 \times \cdots \times \sigma_r)$$

Steenrod operations

Let $\mathcal{P}$ be an operad and let V be a dg-module. Recall that the free $\mathcal{P}$-algebra on V is

$$\mathcal{P}(V) = \bigoplus_{i=1}^{\infty} \mathcal{P}(i) \otimes^{\mathbb{S}_i} V^{\otimes i}$$

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When working in finite characteristic, the cohomology of the free E_∞-algebra is not the symmetric algebra. Instead one has

$$H^\bullet \mathcal{E}(V) = \mathcal{A}(H^\bullet(V))$$

Here $\mathcal{A}$ is the (unstable) Steenrod algebra which contains $\mathrm{Sym}(H^\bullet(V))$

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$$H^\bullet \mathcal{E}(V) = \mathcal{A}(H^\bullet(V))$$

Here $\mathcal{A}$ is the (unstable) Steenrod algebra which contains $\mathrm{Sym}(H^\bullet(V))$ but also extra elements like $\mathrm{Sq}^n(v)$. One has a map

$$\mathcal{A}(H^\bullet(V)) \xrightarrow{H^\bullet(\gamma)} H^\bullet(V)$$

This means that the cohomology of an E_∞-algebra is commutative but also acted on by these extra elements in the Steenrod algebra.

Part 1: Strictly commutative dg-algebras

Motivations

The geometric motivation The starting observation of rational homotopy theory is that, in zero characteristic, every E_∞-algebra is weakly equivalent to a commutative dg-algebra. This viewpoint allows us to "completely" understand spaces rationally.

- A natural question: when does an E_∞-algebra admit a commutative model over $\mathbb{F}_p$ or $\widehat{\mathbb{Z}_p}$?

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- A natural question: when does an E_∞-algebra admit a commutative model over $\mathbb{F}_p$ or $\widehat{\mathbb{Z}_p}$?
- In situations where you cannot give such a model, what is the best model that you can give? What information can we extract from it?

The algebraic motivation Studying E_∞-algebras is hard. There are still being papers written on the primary Steenrod operations and the secondary Steenrod operations are incredibly complicated. Studying commutative dg-algebras gives us insight into this difficult structure in a baby case.

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- Firstly, one can take coinvariants: $\mathcal{P}(A) = \bigoplus_{k=1}^{\infty}(\mathcal{P}(k) \otimes A^{\otimes k})_{\mathbb{S}_k}$. Algebras over this monad are dg-modules A equipped with a binary multiplication $m : A^\bullet \otimes A^\bullet \to A^\bullet$.

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- Secondly, one can take invariants $\Gamma\mathcal{P}(A) = \bigoplus_{k=1}^{\infty} (\mathcal{P}(k) \otimes A^{\otimes k})^{\mathbb{S}_k}$. Algebras over this monad are *divided power algebras*: dg-modules A equipped with a binary multiplication $m : A^\bullet \otimes A^\bullet \to A^\bullet$ and extra operations γ_k which behave like $\frac{x^k}{k!}$. Over $\mathbb{F}_p$, this implies that $x^p = 0$.

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- Finally one has a monad $\mathcal{P}(A) \to \Lambda\mathcal{P}(A) \to \Gamma\mathcal{P}(A)$ given by the image of the norm map.

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When we are working over a field of characteristic 0 (the classical theory of Loday-Vallette) or the action of $\mathbb{S}_k$ on $\mathcal{P}(k)$ is free (theory of quasi-planar operads of Le Grignou-Roca Lucio), invariants coincide with coinvariants and the three notions above coincide (subject to certain finiteness assumptions).

Model structures

Theorem (Hinich, 1997)

Let $\mathcal{P}$ be a cofibrant (or $\mathbb{S}$-split) operad over a commutative ring R. Then the category of $\mathcal{P}$-algebras over R is a closed model category with quasi-isomorphisms as the weak equivalences and surjective maps as fibrations.

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Example

Consider $M = \mathbb{F}_p[x \to dx]$. One has $H^\bullet(\mathrm{Sym}(M)) \neq 0$ because 1) x^{p^n} is a cocycle 2) $x^{p^n - 1}dx$ is not closed.

Part 2: Obstruction theory over $\mathbb{F}_p$

A crash-course in Massey products 1

When is a commutative algebra over $\mathbb{Q}$ weakly equivalent to its cohomology?

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Definition

Let A be a dg-algebra. Let $a, b, c \in H^\bullet(A)$ by such that $ab = 0$ and $bc = 0$. Let x, y, z be cocycles representing a, b, c and suppose $du = xy$ and $dv = yz$. Then $uz - xv$ is a cocycle that we call the (primitive, secondary) Massey product, it represents a well-defined class of

$$\frac{H^{|a|+|b|+|c|-1}(A)}{aH^{|b|+|c|-1}(A) + H^{|a|+|b|-1}(A)c}$$

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$$\frac{H^{|a|+|b|+|c|-1}(A)}{aH^{|b|+|c|-1}(A) + H^{|a|+|b|-1}(A)c}$$

Proposition (Massey, 1958)

If for some $a, b, c \in H^\bullet(A)$, the class above is nonzero, then A is not formal

Higher Massey products

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- More recently, this machinery for primitive Massey products has been extended to general quadratic operads. (Muro /FC.-Moreno-Fernandez)

Coherent vanishing of Massey products

Unfortunately, it is not enough for Eilenberg-Moore spectral sequence to collapse on E_2-page, in other words, for all Massey products to vanish. Formality turns out to be equivalent to all of these vanishing in a *coherent* way.

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Theorem (Deligne, Griffiths, Morgan, Sullivan, 1975)

Let A be a commutative dg-algebra in $\mathbb{Q}$-vector spaces. Let $\mathfrak{m} = (\mathrm{Sym}(\bigoplus_{i=0}^{\infty} V_i), d)$ be the minimal model for A. Then A is formal if and only if, there is in each V_i a complement B_i to the cocycles Z_i, $V_i = Z_i \oplus B_i$, such that any closed form, a, in the ideal, $I(\bigoplus_{i=0}^{\infty} B_i)$, is exact.

Sullivan algebras

Definition

Let $\mathcal{P}$ be an operad over a field and A is a $\mathcal{P}$-algebra. A *Sullivan model* for A is a semi-free algebra $f : (\mathcal{P}(\bigoplus_{i=0}^{\infty} V_i), d) \xrightarrow{\sim} A$ such that

- the map $f|_{V_0} : V_0 \to A$ is a weak equivalence of dg-vector spaces. In particular $V_0 = H^\bullet(A)$.
- the differential satisfies $d(V_k) \subseteq (\mathcal{P}(\bigoplus_{i=0}^{k-1} V_i), d)$.
- We require that $V_k \oplus (\mathcal{P}(\bigoplus_{i=0}^{k-1} V_i) \to A$ is a weak equivalence for each k.

The intuition is that a Sullivan model captures the idea of building a quasi-free resolution in stages, starting with a map $H \to A$ and progressively killing cocycles.

$\mathcal{P}$-Massey products

We use truncated Sullivan algebras to define cotriple products in this context.

Definition

A *N-step Sullivan model* for A is a semi-free algebra $f : (\mathcal{P}(\bigoplus_{i=0}^{N} V_i), d) \to A$ such that

- the differential satisfies $d(V_k) \subseteq (\mathcal{P}(\bigoplus_{i=0}^{k-1} V_i), d)$
- We require that $V_k \oplus (\mathcal{P}(\bigoplus_{i=0}^{k-1} V_i) \to A$ is a weak equivalence for each k.

Let $I(\mathcal{P}(\bigoplus_{i=1}^{N} V_i), d))$ be the ideal generated by $\mathcal{P}(\bigoplus_{i=1}^{k-1} V_i), d)$. We call nonzero $\sigma \in H^\bullet(I(\mathcal{P}(\bigoplus_{i=1}^{k-1} V_i), d))$ an N^{th} order cotriple product with value $H^\bullet(f)(\sigma)$

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Homotopy invariance

Theorem

Let $\mathcal{P}$ be an operad that reflects homotopy equivalences. A morphism of $\mathcal{P}$-algebras $f : A \to B$ preserves cotriple product sets. If furthermore f is a quasi-isomorphism, then $H^*(f)$ induces a bijection between the corresponding cotriple product sets.

Spectral sequence interpretation

Theorem

Let $\mathcal{P}$ be an operad that reflects homotopy equivalences and A be a $\mathcal{P}$-algebra. Take the cotriple resolution of A and consider the spectral sequence obtained from the skeletal filtration. Then cotriple products represent the differentials in this spectral sequence.

Massey products in positive characteristic

Over $\mathbb{F}_p$ there are more secondary operations.

Definition (F. C.)

Let A be a commutative dg-algebra over $\mathbb{F}_p$. Let $x, y \in H^\bullet(A)$ be such that $xy = 0$. Choose cocycles $a, b \in A$ representing x, y respectively. Then there exists $c \in A$ such that $dc = xy$. Then c^p is a cocycle which we call the *type 1 secondary commutative product* of x and y. This represents a well defined element of

$$\frac{H^{p(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^p + x^p H^{p(|y|-1)}(A) + y^p H^{p(|x|-1)}(A)}$$

where the term $x^p H^{p(|y|-1)}(A) + y^p H^{p(|x|-1)}(A)$ in the denominator accounts for the choice of representatives x and y.

Type 2 commutative products

Definition (F. C.)

Let p be an odd prime. Then there is a *type 2 secondary commutative product* defined for $x, y \in H^*(A)$ such that $xy = 0$ we choose cocycles $a, b \in A$ representing x, y respectively. Then there exists $c \in A$ such that $dc = xy$. Then $c^{p^n-1}ab$ is a cocycle which we call the *type 2 secondary commutative product* of x and y. In this case, the operation represents a well-defined element of

$$\frac{H^{p^n(|x|+|y|-1)+|x|+|y|}(A)}{H^{(|x|+|y|-1)}(A)^{p^n-1} \cdot xy}$$

Observe that $d(\frac{1}{p}c^p) = c^{p-1}ab$. Therefore type 2 secondary commutative products vanish on divided power algebras. Therefore this kind of operation provides an obstruction for a commutative algebra A to be weakly equivalent to a divided power algebra.

Completeness of secondary operations

Definition

We call a cotriple product *primitive* if it arises from monomial relations in cohomology.

Proposition

All secondary primitive cotriple products on a commutative dg-algebra A over $\mathbb{F}_p$ are linear combinations of

- *classical Massey products.*
- *Type 1 secondary commutative operations*
- *Type 2 secondary commutative operations.*

Secondary cotriple operations: Producing counterexamples

Cotriple products can be used to produce examples of:

- Using type 1 operations. Commutative algebras A, B over $\mathbb{Z}$ such that $A \otimes \mathbb{Q}$ and $B \otimes \mathbb{Q}$ are weakly equivalent, but $A \otimes \mathbb{F}_p$ and $B \otimes \mathbb{F}_p$ are not.
- Using type 2 operations. Commutative algebras which have a divided power structure on cohomology but which are not weakly equivalent to a divided power algebra.

Applications of secondary cotriple operations: Producing counterexamples

The extra cotriple operations are preserved by maps of commutative algebras but not associative algebras. So one has:

- Commutative algebras A, B over $\mathbb{F}_p$, which are weakly equivalent as associative algebras but not commutative algebras.

Finally, studying the indeterminacies of third order cotriple products, one can produce examples of:

- Commutative algebras A, B over $\mathbb{F}_p$ that are weakly equivalent as E_∞-algebras but not commutative algebras.

Obstructions to strict commutativity

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The E_∞-algebra $C^\bullet(X, \mathbb{F}_p)$ is rectifiable iff X is the disjoint union of contractible spaces.

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Proposition (Mandell, 2009)

The E_∞-algebra $C^\bullet(X, \mathbb{F}_p)$ is rectifiable iff X is the disjoint union of contractible spaces.

There are less obvious obstructions given by secondary operations.

Conjecture (Mandell, 2009)

Let X be a finite n-connected simplicial set. Then, after inverting finitely many primes $C^\bullet(X, \mathbb{Z})$ has a commutative model as an E_n-algebra. If X is formal, then, after possibly inverting more primes, this commutative model is formal.

Coherent vanishing of higher Steenrod operations

Definition

Let A be an E_∞-algebra over $\mathbb{F}_p$. Then the higher Steenrod operations *vanish coherently* if for every (or any) Sullivan resolution $(\mathcal{E}(\bigoplus_{i=0}^\infty V_i), d)$ for A, there exists a splitting $V_i = X_i \bigoplus Y_i$, with $X_0 = V_0$; such that $(\mathrm{Sym}(\bigoplus_{i=0}^\infty X_i), d)$ is a Sullivan algebra and the kernel of

$$(\mathcal{E}(\bigoplus_{i=0}^{\infty} V_i), d) \to (\mathrm{Sym}(\bigoplus_{i=0}^{\infty} X_i), d)$$

is acyclic.

Coherent vanishing

Theorem (FC)

Let A be an E_∞-algebra over $\mathbb{F}_p$. Then A is rectifiable if and only if its higher Steenrod operations vanish coherently.

Similar result for formality.

Part 2: de Rham forms in positive characteristic

Motivation: we want to find a best commutative approximation to the E_∞ algebra $C^\bullet(X, \widehat{\mathbb{Z}_p})$.

de Rham forms in positive characteristic

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Key idea: We want to imitate Sullivan's approach to rational homotopy theory.

Theorem (Sullivan, 1978)

Suppose one has a functor $A_{PL} : \triangle^\bullet \to \mathrm{CDGA}_{\mathbb{Q}}$ that satisfies the Poincaré Lemma: $H^0(\triangle^n, \mathbb{Q}) = \mathbb{Q}$ and $H^i(\triangle^n, \mathbb{Q}) = 0$ for $i > 0$; and which is extendable $\pi_k(A^k_{PL}(\triangle^\bullet)) = 0$ for all $k \geq 0$. Then the left Kan extension along $\triangle^\bullet \to \mathrm{Set}_\triangle$

$$A_{PL} : \mathrm{Set}_\triangle \to \mathrm{CDGA}_{\mathbb{Q}}$$

is such that there is a zig-zag of E_∞-algebras

$$A^\bullet_{PL}(X) \xrightarrow{\sim} (A_{PL} \otimes C)^\bullet(X) \xleftarrow{\sim} C^\bullet(X, \mathbb{Q})$$

Part 2: de Rham forms in positive characteristic

Definition

The simplicial cochain coalgebra $\Omega^*_\bullet$ has for n-simplices

$$\Omega^*_n = \frac{\widehat{\mathbb{Z}_p}\langle x_0, \ldots x_n\rangle \otimes \Lambda(dx_0, \ldots, dx_n)}{(x_0 + \cdots + x_n - p, dx_0 + \cdots dx_n)}, \quad |x_i| = 0, \quad |dx_i| = 1.$$

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The differential $d : \Omega^*_n \to \Omega^{*+1}_n$ is determined by the formula

$$d(f) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i$$

for $f \in \Gamma_p(x_0, \ldots, x_n)/(x_0 + \cdots + x_n - p)$ and then extended by the Leibniz rul

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The simplicial cochain coalgebra $\Omega^*_\bullet$ has for n-simplices

$$\Omega^*_n = \frac{\widehat{\mathbb{Z}_p}\langle x_0, \ldots x_n \rangle \otimes \Lambda(dx_0, \ldots, dx_n)}{(x_0 + \cdots + x_n - p, dx_0 + \cdots dx_n)}, \quad |x_i| = 0, \quad |dx_i| = 1.$$

The differential $d : \Omega^*_n \to \Omega^{*+1}_n$ is determined by the formula

$$d(f) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i$$

for $f \in \Gamma_p(x_0, \ldots, x_n)/(x_0 + \cdots + x_n - p)$ and then extended by the Leibniz rul . The simplicial structure is defined as follows

$$d_i^n : \Omega^*_n \to \Omega^*_{n+1} : x_k \mapsto \begin{cases} x_k & \text{for } k < i. \\ 0 & \text{for } k = i. \\ x_{k-1} & \text{for } k > i. \end{cases}$$

and

$$s_i^n : \Omega^*_n \to \Omega^*_{n-1} : x_k \mapsto \begin{cases} x_k & \text{for } k < i. \\ x_k + x_{k+1} & \text{for } k = i. \\ x_{k+1} & \text{for } k > i. \end{cases}$$

The cohomology of de Rham forms

Cartan considered a similar construction except over $\mathbb{Z}\langle t\rangle$. The functor

$$\Omega : \triangle^{\bullet} \to \mathsf{CDGA}_{\widehat{\mathbb{Z}_p}}$$

satisfies the Poincaré Lemma but is not extendable.

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satisfies the Poincaré Lemma but is not extendable. However, it is *almost extendable* and one can suitably modify Sullivan's proof to produce the following.

Theorem (Cartan, F.C)

Consider the left Kan extension along $\triangle^{\bullet} \to \mathsf{Set}_{\triangle}$

$$\Omega : \mathsf{Set}_{\triangle} \to \mathsf{CDGA}_{\widehat{\mathbb{Z}_p}}$$

Then there is an isomorphism of cohomology algebras

$$H^{\bullet}(X, \widehat{\mathbb{Z}_p}) = H^{\bullet}(\Omega(X)).$$

The homotopy type of the de Rham forms

What about the E_∞-homotopy type?

Definition

Let X be a simplicial set. We define the *altered singular cochain algebra* $\mathcal{C}^\bullet(X)$ to be the following subalgebra of the singular cochains $C^\bullet(X)$.

$$\mathcal{C}^n(X) = \left\langle p^i\sigma : \text{ for } \sigma \in C^n(X, \widehat{\mathbb{Z}_p}) \text{ and } \begin{cases} i = n & \text{if } d\sigma = 0. \\ i = n+1 & \text{otherwise.} \end{cases} \right\rangle$$

The differential and the E_∞ structure are that induced by those on $C^\bullet(X, \widehat{\mathbb{Z}_p})$.

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Theorem (F.C.)

As an E_∞-algebra, $\Omega(X)$ is quasi-isomorphic to $\mathcal{C}(X)$.

Connection with crystalline cohomology

When X is a scheme: the altered singular cochain algebra can also be interpreted as

$$\mathcal{C}(X) = \eta(C^\bullet(X, \widehat{\mathbb{Z}_p}))$$

where η is the Berthelot-Ogus-Deligne *décalage* functor, which is defined as the connective cover with respect to the Beilinson t-structure on filtered complexes with repect to the p-adic filtration. The rectifiability of this algebra recovers a result of Bhatt-Lurie-Mathew.

Massey products and $\Omega(X)$

The model can be used to compute Massey products.

Proposition (F.C.)

Suppose that $\sigma \in H^\bullet(X, \mathbb{Q})$ *be the higher Massey product of* $\langle x_1, x_2, \ldots, x_n \rangle \in H^\bullet(A_{PL}(X), \mathbb{Q})$. *Then there exists an* $n > 0$ *such that* $p^n\sigma \in H^\bullet(X, \widehat{\mathbb{Z}_p})$ *is the higher Massey product of* $\langle p^n x_1, p^n x_2, \ldots, p^n x_n \rangle \in H^\bullet(A_{PL}(X), \widehat{\mathbb{Z}_p})$ *computed in* $\Omega^\bullet(X)$.

Similarly it can also be compute Massey products in the torsion part of the cohomology.

Formality of $\Omega(X)$

Finally, we have this theorem which is inspired by Mandell's conjecture.

Theorem (F.C.)

Let X be a finite simplicial set such that $A_{PL}(X)$ is formal over $\mathbb{Q}$. For all but finitely many primes, $\Omega^\bullet(X)$ is formal over $\widehat{\mathbb{Z}_p}$ as a dg-commutative algebra.

Towards a proof of Mandell's conjecture

I propose the following proof sketch of Mandell's conjecture:

- First, observe that if X is a finite n-connected simplicial set then it admits a finite n-reduced model X' such that then $\mathcal{E}_n(N)$ acts trivially on $C^*(X', \mathbb{Z})$ for $N >> 0$ for degree reasons.
- Invert all primes $p < N$.
- Conjecture: As an E_n-algebra, $C^*(X', \mathbb{Z}_{(p)})$ is weakly equivalent to $C^*(X'_{\mathbb{Q}}, \mathbb{Z}_{(p)})$.
- Construct a functorial commutative model for $C^*(X, \mathbb{Q})$ by piecing together commutative models for $C^*(\triangle^m/\{\text{n-skeleton}\}, \mathbb{Z}_{(q)})$

Further questions

- Is there a notion of localisation on the category of topological spaces that models equivalence of de Rham forms?
- Does the divided power homotopy type determine the associative homotopy type?
- Are two divided power algebras quasi-isomorphic if and only of they are quasi-isomorphic as associaive algebras?
- Is there Koszul duality between restricted Lie algebras and divided powers algebras?
- Is there a formulation of the coherent vanishing theorem in terms of Hochschild cohomology and Kaledin-like classes?